ANTICHAIN SUBSET SUMS IN RANK-THREE MULTIPLICATIVE SEMIGROUPS

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AI disclosure. The argument presented in this paper was found using the Principia Math Harness. The human authors subsequently expanded the argument.

Abstract. Let $a, b, c \geq 2$ be pairwise coprime and let $S = \{a^k b^\ell c^m : k, \ell, m \geq 0\}$. For every fixed real $1 < \rho < \min(a, b, c)$, every sufficiently large integer is a subset sum of $S \cap [x, \rho x)$ for some x. Such a subset is automatically a divisibility antichain. The proof uses arithmetic rigidity on a triangular grid in exponent space and a local central limit theorem. We give a quantitative convergence rate, a local theorem for tilted Bernoulli laws, asymptotic representation counts at linear scales, and a represented interval with endpoints $O(x \log x \log\log x)$ from the ends of the total-weight interval. An entropy bound locates the saturation threshold up to a factor of order $(\log\log x)^2$. We also prove a joint local theorem for the sum and the number of summands, identifying the exact lattice on which the pair is supported. Exact congruence identities, fixed-denominator well asymptotics, archimedean harmonics, the rank threshold, and a qualitative shrinking-band consequence complete the picture; a finite-well classification and quantitative shrinking rates are left open.

1. Statement and reduction

Fix pairwise-coprime integers $a, b, c \geq 2$, and write

$$S = \{a^k b^\ell c^m : k, \ell, m \in \mathbb{Z}_{\geq 0}\}, \qquad d = \min(a, b, c).$$

Unless stated otherwise, fix $\rho = p/q \in \mathbb{Q}$ with $1 < \rho < d$, where $p, q \in \mathbb{N}$. For an integer $x \geq 1$, set

$$S_x = S \cap [x, \rho x), \quad m_x = |S_x|, \quad W_x = \sum_{s \in S_x} s, \quad V_x^2 = \sum_{s \in S_x} s^2, \qquad \mu_x = \frac{W_x}{2}, \quad \sigma_x = \frac{V_x}{2}.$$

Let $(\xi_s)_{s \in S_x}$ be independent fair Bernoulli variables, and put $Y_x = \sum_{s \in S_x} s\xi_s$.

Theorem 1.1 (Local limit theorem)**.** *As $x \to \infty$ through the positive integers,*

$$\mathbb{P}(Y_x = n) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left(-\frac{(n - \mu_x)^2}{2\sigma_x^2}\right) + o(\sigma_x^{-1}), \tag{1}$$

uniformly for $n \in \mathbb{Z}$. In particular, for all sufficiently large x, every integer with $|n - \mu_x| \leq \sigma_x$ is a subset sum of S_x.

Theorem 1.2 (Erdős Problem 123)**.** *For every real $1 < \rho < d$, every sufficiently large integer N can be written as*

$$N = \sum_{s \in A} s, \qquad A \subset S \cap [x, \rho x)$$

for some positive integer x. No two distinct elements of A divide one another.

The last assertion follows directly from coprimality. If $u < v$ belong to S and $u \mid v$, comparison of the prime valuations in the three pairwise-coprime bases shows that $v/u = a^r b^s c^t \geq d$ with $r, s, t \geq 0$. Two elements of $[x, \rho x)$ have ratio less than $\rho < d$. A real ratio is handled by first choosing a rational $1 < \rho' < \rho$.

For $(a, b, c) = (2, 3, 5)$, Theorem 1.2 answers both narrow-band questions in Erdős [E92, §21, p. 239]: for every fixed $\varepsilon > 0$, every sufficiently large integer is a sum of distinct terms whose largest member is

less than $(1+\varepsilon)$ times the smallest. Indeed, choose $1 < \rho < \min(2, 1+\varepsilon)$ and apply the theorem. The assertion with ratio two follows as well.

Historical context and related work. A sequence is called *d-complete* if every sufficiently large integer is a sum of distinct terms from the sequence, no one of which divides another; this terminology was introduced by Erdős and Lewin [EL96]. The band requirement in Theorem 1.2 is stronger than d-completeness. In his 1992 problem collection, Erdős [E92, §21, p. 239] asked whether sums of distinct $2,3,5$-smooth integers could always be chosen with largest summand less than twice the smallest, and proposed the stronger ratio $1+\varepsilon$ for each fixed $\varepsilon > 0$ and all sufficiently large targets. The same discussion records the elementary inductive solution, due to Jansen and others, of the unrestricted antichain problem for $\{2^k 3^\ell\}$.

Unrestricted completeness is classical. Birch proved that, for coprime integers $a, b > 1$, every sufficiently large integer is a sum of distinct elements of $\{a^k b^\ell : k, \ell \geq 0\}$ [B59]; hence the same is true after adjoining a third base. Cassels generalized this completeness result by an analytic criterion [C60]. Neither imposes the single-band restriction studied here.

Erdős and Lewin [EL96, p. 840] formulated the general conjecture for three pairwise-coprime bases. They established, among other cases, d-completeness for $(3,5,7)$ and for $(2,5,p)$ with $p \in \{7,11,13,17,19\}$. Ma and Chen [MC16] subsequently gave a finite-interval criterion for the family $(2,5,p)$ and proved the cases $p \in \{9,21,23,27,29,31\}$. Further Erdős–Lewin conjectures concerning d-complete sequences were studied by Yu and Chen [YC22]. Chen and Yu [CY23] then established a criterion for general pairwise-coprime triples, with applications to $(3,5,r)$ for $1 < r \leq 14$, $(2,5,r)$ for $1 < r \leq 87$, and $(2,7,r)$ for $1 < r \leq 33$, in each case subject to pairwise coprimality.

The two-base band question has a separate history. Erdős and Lewin [EL96, p. 838] gave a counting obstruction for $2,3$-smooth summands in $(x, 2x)$ and asked whether a larger fixed ratio suffices. Van Doorn and Everts [DE25] proved that every positive integer is a sum of distinct $2,3$-smooth integers whose largest term is less than six times the smallest. Their conclusion does not require a divisibility antichain and does not give arbitrarily narrow bands. It is therefore compatible with the rank-two obstruction in Section 13.

Colin Snyder proved the unrestricted rank-three antichain conjecture in July 2026, before the Fourier argument presented here. His proof uses van der Waerden's theorem and radix induction on a homogeneous exponent level; a public Lean development is available [SF26], and the Formal Conjectures entry [FC26] records the solution. We claim no priority for unrestricted d-completeness. Our contribution is the local limit theorem for each sufficiently large fixed band, its arbitrarily narrow fixed-band representation consequence, the single-band saturation bounds, and the joint local theorem for the sum and cardinality.

The extensions in Sections 5–14 develop directions first explored in [H26]. Three items from that note, a finite-well classification, quantitative shrinking rates, and an explicit numerical threshold, are stated below as open problems and are not used in any proof. All implied constants may depend on the fixed bases, the fixed band ratio, and other parameters explicitly held fixed. Write

$$\alpha = \log a, \quad \beta = \log b, \quad \gamma = \log c, \quad \eta = \log \rho, \quad L = \log x.$$

Strategy. The band contains $\asymp L^2$ elements, each of size $\asymp x$. Near zero, its Bernoulli characteristic function has the expected Gaussian approximation. The issue is to control the rest of the Fourier integral at the smaller scale $1/(xL)$. We construct a triangular grid of band elements with bounded multiplicative relations along its edges. Good edges transport nearest integers exactly. A row and a transversal path meet all three coordinate faces; the collective gcd of their endpoint weights is one. Selecting these paths despite an energy-budgeted defect set, and counting the possible edge defects, gives the sublevel estimate (9). Its factor x^{-1} is essential in the ensuing layer integral: a positive minor-arc energy floor alone would not give the same estimate. Fourier inversion then proves the local theorem, and a bounded-increment sweep of W_x supplies every sufficiently large target.

Organization. Sections 2–4 prove Theorems 1.1 and 1.2. Sections 5–6 sharpen the local theorem: a convergence rate, a uniform theorem for tilted Bernoulli laws, and from it the full linear window and

asymptotic representation counts. Section 7 determines where a single band's represented set saturates, up to a factor $(\log\log x)^2$, and Section 11 explains why the present sublevel estimate stops there. Section 10 controls the sum and the number of summands jointly. Sections 8–9 record exact identities at rational frequencies and their consequences for residue biases; the finite-well classification is left as Conjecture 8.4 and the pointwise correction as Question 9.2. Sections 12–14 treat shrinking bands, the rank threshold, non-coprime bases, and effectivity.

2. A triangular grid in the exponent slab

The set

$$E_L = \{(k,\ell,m) \in \mathbb{Z}^3_{\geq 0} : L \leq \alpha k + \beta\ell + \gamma m < L + \eta\}$$

parametrizes S_x. Ratios of distinct members of $\{\alpha,\beta,\gamma\}$ are irrational: a rational equality $\alpha/\beta = r/s$ would imply $a^s = b^r$, contrary to coprimality.

We first record a rounding consequence of irrational rotation. If $u, v > 0$, $u/v \notin \mathbb{Q}$, and $0 < \eta < v$, then there is R such that whenever

$$uX + vY = A + \eta/2, \qquad X, Y \geq R,$$

there are nonnegative integers r, s, each within R of its corresponding real coordinate, with

$$A \leq ur + vs < A + \eta. \tag{2}$$

Indeed, finitely many multiples of u/v form an $\eta/(8v)$-net modulo one. Set $r_0 = \lfloor X \rfloor$ and $y = (A - ur_0)/v$. Choose a bounded integer h so that the fractional part of $y - hu/v$ lies between $1 - 5\eta/(8v)$ and $1 - 3\eta/(8v)$, and take $r = r_0 + h$, $s = \lceil y - hu/v \rceil$. This gives (2); the linear relation bounds the error in the second coordinate.

Applying this argument to the three pairs of logarithms gives R_0 such that every point on

$$\alpha X + \beta Y + \gamma Z = L + \eta/2 \tag{3}$$

whose positive coordinates are at least $2R_0$, and of which at most one is zero, lies within R_0 coordinatewise of a point of E_L. A zero coordinate remains zero. If all coordinates are positive, round Y to an integer ℓ_0, replace Z by $Z + (\beta/\gamma)(Y - \ell_0)$, and apply (2) to α, γ. Increasing R_0 absorbs these bounded preliminary changes.

Proposition 2.1 (Geometry and moments)**.** *There are positive constants c, C and integers $D, K \geq 1$ such that, for all large x,*

$$cL^2 \leq m_x \leq CL^2, \qquad cxL \leq V_x \leq CxL, \qquad cxL^2 \leq W_x \leq CxL^2. \tag{4}$$

Moreover, for some $n \asymp L$ there are distinct $s_v \in S_x$ indexed by

$$\Delta^\circ_n = \{(i,j,r) \in \mathbb{Z}^3_{\geq 0} : i + j + r = n\} \setminus \{(n,0,0),(0,n,0),(0,0,n)\},$$

such that a zero coordinate of v gives a zero corresponding exponent of s_v, and adjacent vertices satisfy

$$A_{vw} s_v = B_{vw} s_w, \qquad 1 \leq A_{vw}, B_{vw} \leq K. \tag{5}$$

Proof. Choose τ_0 so large that $\tau_0/\max(\alpha,\beta,\gamma) > 4R_0$. Set

$$n = \left\lfloor \frac{L + \eta/2}{\tau_0} \right\rfloor, \qquad \tau = \frac{L + \eta/2}{n}.$$

For large L, $n \asymp L$ and $\tau_0 \leq \tau \leq 2\tau_0$. For $v = (i,j,r)$ round the point $(\tau i/\alpha, \tau j/\beta, \tau r/\gamma)$ to E_L. The omitted corners are precisely the points at which the rounding statement is unavailable. Distinct targets have a coordinate separated by more than $4R_0$, so the images are distinct. Adjacent rounded exponent triples differ by at most a fixed D in each coordinate. Clearing positive and negative exponent differences gives (5) with $K = (abc)^D$.

There are $\asymp n^2$ vertices, giving the lower bound on m_x. Conversely, $k, \ell \ll L$, and for each pair there is at most one m, since $\eta < \gamma$. Thus $m_x \ll L^2$. The remaining estimates follow from $x \leq s < \rho x$. The construction also works for real x. □

The later sections need the asymptotic form of these moments.

Lemma 2.2 (Weighted band asymptotics)**.** *For every fixed Riemann-integrable function f on $[0,\eta]$,*

$$\sum_{s\in S_x} f(\log(s/x)) = \frac{L^2}{2\alpha\beta\gamma}\int_0^\eta f(u)\,\mathrm{d}u + o(L^2). \tag{6}$$

In particular,

$$m_x \sim \frac{\eta L^2}{2\alpha\beta\gamma}, \quad W_x \sim \frac{(\rho-1)xL^2}{2\alpha\beta\gamma}, \quad V_x^2 \sim \frac{(\rho^2-1)x^2L^2}{4\alpha\beta\gamma}. \tag{7}$$

On any fixed coordinate face, the corresponding two-base weighted count is

$$\sum_{\substack{k,\ell\ge 0\\ B\le \alpha k+\beta\ell < B+\eta}} f(\alpha k+\beta\ell - B) = \frac{B}{\alpha\beta}\int_0^\eta f(u)\,\mathrm{d}u + o(B). \tag{8}$$

Proof. For an irrational rotation, averages of a fixed Riemann-integrable function converge to its integral uniformly in the starting phase. For completeness, for a nonconstant trigonometric monomial the finite geometric-sum formula bounds the average by $O(1/M)$, uniformly in that phase. Approximation by trigonometric polynomials proves the assertion for continuous functions; upper and lower continuous approximations prove it for Riemann-integrable functions.

For $0 \le k \le B/\alpha$, put $y_k = (B-\alpha k)/\beta$. Since $\eta < \beta$, there is at most one allowable ℓ, and its excess above B is $\beta(\lceil y_k\rceil - y_k)$. Apply the preceding rotation assertion to the periodic function which equals $f(u)$ on $0 \le u < \eta$ and is zero on the rest of $[0,\beta)$. The $O(1)$ choices at the endpoint $k = B/\alpha$ do not affect the result. This proves (8).

Now sum (8) with $B = L - \gamma m$. The main terms sum to

$$\frac{1}{\alpha\beta}\int_0^\eta f(u)\,\mathrm{d}u \sum_{0\le m\le L/\gamma}(L-\gamma m) = \frac{L^2}{2\alpha\beta\gamma}\int_0^\eta f(u)\,\mathrm{d}u + O(L).$$

The errors sum to $o(L^2)$: first bound each error by εB for B above a fixed threshold, then handle the bounded terminal range separately. Taking $f(u) = 1, e^u, e^{2u}$ gives (7). □

Corollary 2.3 (Fixed exponent residue classes)**.** *For a fixed integer $h \ge 1$, prescribing the three exponents modulo h gives $m_x/h^3 + o(L^2)$ elements of S_x. Prescribing two exponents modulo h on a fixed coordinate face gives h^{-2} times that face's leading count, with error $o(L)$. The same statements hold after imposing fixed lower bounds on the free exponents.*

Proof. Write each exponent as its prescribed residue plus a multiple of h, and apply Lemma 2.2 to the bases raised to the hth power and to a band whose lower endpoint is changed by a fixed multiplicative factor. Fixed lower bounds can be absorbed by another such shift. □

3. Low-energy frequencies

For $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, write

$$Q_x(t) = \sum_{s\in S_x} \|st\|^2, \qquad \|y\| = \min_{m\in\mathbb{Z}} |y-m|.$$

Proposition 3.1 (Rigidity)**.** *There are constants $z_0, C, \kappa, \delta > 0$ such that, for all large x and $0 \le z \le z_0L^2$,*

$$\operatorname{meas}\{t\in\mathbb{T} : Q_x(t) \le z\} \le \frac{1}{x}\exp\!\left(C\left(1+\frac{z}{L}\right)\log(L+2)\right). \tag{9}$$

The constants can be chosen with $\delta d \le 1/8$ and

$$Q_x(t) < \kappa L \quad\Longrightarrow\quad \|t\| \le \delta/x. \tag{10}$$

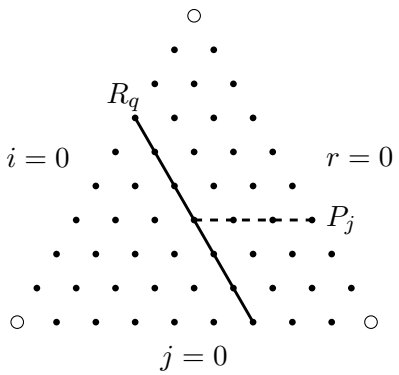


FIGURE 1. The punctured triangle Δ_n° ($n = 9$; corners removed). A row R_q (solid) runs between the faces $i = 0$ and $j = 0$; a path P_j (dashed) leaves the row and reaches the third face $r = 0$. Their union is connected and touches all three faces, which is what forces the constant of proportionality in Step 5 of the proof of Proposition 3.1 to be an integer.

In fact, any fixed smaller positive δ is permissible after changing the other constants and the threshold for x.

Proof. Step 1: energy pays for bad vertices. Use the grid of Proposition 2.1, and choose $0 < \delta_0 < \min(1/(4K), 1/32)$. Let m_v be a nearest integer to $s_v t$, and call v bad if $|s_v t - m_v| > \delta_0$. If H vertices are bad, then

$$H\delta_0^2 \le Q_x(t). \tag{11}$$

Step 2: a row and a path with few defects (Figure 1). Among the disjoint rows

$$R_q = \{(i, q-i, n-q) : 0 \le i \le q\}, \qquad n/3 \le q \le 2n/3,$$

one has at most $4H/n$ bad vertices. From its central half consider the disjoint paths

$$P_j = \{(q-j+r, j, n-q-r) : 0 \le r \le n-q\}, \qquad q/4 \le j \le 3q/4.$$

One has at most $7H/n$ bad vertices. The row and path form a connected tree with endpoints on the three coordinate faces and only $O(1 + H/n)$ bad vertices. If $8H < n$, both can be chosen completely good.

Step 3: integer transport along good edges. Orient its edges and put $e_{vw} = B_{vw}m_w - A_{vw}m_v$. For a good edge,

$$|e_{vw}| \le B_{vw}|m_w - s_w t| + A_{vw}|m_v - s_v t| < 2K\delta_0 < 1,$$

so $e_{vw} = 0$. Every edge value, good or not, has absolute value at most K.

Step 4: encoding the defective edges. If $Q_x(t) \le z$, at most $O(1 + z/L)$ edge values are nonzero. The row, path, positions of these values, and their values have at most

$$\exp\left(C\left(1 + \frac{z}{L}\right)\log(L+2)\right) \tag{12}$$

possible descriptions.

Step 5: decoding a fixed code. For fixed data, compare two compatible nearest-integer vectors. Their difference satisfies

$$B_{vw}\Delta_w = A_{vw}\Delta_v, \qquad B_{vw}s_w = A_{vw}s_v,$$

so connectivity gives $\Delta_v = rs_v$ for a rational r. The three end weights respectively omit a, b, c. Their gcd is one, because the bases are pairwise coprime. Thus $r \in \mathbb{Z}$. Taking $t \in [0, 1)$, a root of weight s_0 has nearest integer between 0 and s_0; fixed data therefore allow only a bounded number of root values, each restricting t to an interval of length $1/s_0 \le 1/x$. This proves (9).

Step 6: very-low-energy localization. If $Q_x(t) < \kappa L$ for sufficiently small κ, then $8H < n$. On a completely good tree all edge values vanish, and m_v/s_v is an integer independent of v. Initially this gives $\|t\| \le \delta_0/x$. Repeat on a fixed narrower band with rational ratio $1 < \rho_0 \le \min(\rho, 3/2)$, decreasing κ if necessary. Writing $t = r + u$, we get $|u| \le 1/(32x)$, so there is no nearest-integer wrap on this narrower band. Hence

$$Q_x(t) \ge u^2 \sum_{s \in S \cap [x, \rho_0 x)} s^2 \gg x^2 L^2 u^2.$$

The assumption $Q_x(t) < \kappa L$ now gives $|u| \ll 1/(x\sqrt{L})$, which is at most δ/x for any fixed positive δ once x is large. □

In particular, $\gcd(S_x) = 1$ for all large x, since the three face weights used in the proof already have gcd one.

4. Fourier analysis and completion of the main proof

We use

$$|\cos(\pi y)| \le e^{-2\|y\|^2}. \tag{13}$$

For $0 \le u \le 1/2$, this follows from $\cos(\pi u) = 1 - 2\sin^2(\pi u/2) \le 1 - 2u^2 \le e^{-2u^2}$.

Lemma 4.1 (Fourier tails)**.** *For $T = L^{1/4}$,*

$$\int_{T/V_x \le |t| \le 1/2} \left| \prod_{s \in S_x} \cos(\pi s t) \right| \mathrm{d}t = o(V_x^{-1}). \tag{14}$$

Proof. For large x, $T/V_x < 2\delta/x$. On $T/V_x \le |t| \le 2\delta/x$, every $|st| < 2\delta d \le 1/4$, so the integrand is at most $e^{-2V_x^2 t^2}$. Its integral is bounded by

$$\frac{2}{V_x} \int_T^\infty e^{-2u^2} \, \mathrm{d}u = o(V_x^{-1}).$$

On $2\delta/x \le |t| \le 1/2$, rigidity gives $Q_x(t) \ge \kappa L$. Layering by Q_x, using (9), bounds the remaining integral by

$$\frac{2}{x} \int_{\kappa L}^{z_0 L^2} e^{-2z + C(1+z/L)\log(L+2)} \, \mathrm{d}z + e^{-2z_0 L^2} = O\Big(x^{-1} L^C e^{-3\kappa L/2}\Big) + e^{-2z_0 L^2}. \tag{15}$$

This is $o((xL)^{-1}) = o(V_x^{-1})$. □

Proof of Theorem 1.1. Fourier inversion gives

$$\mathbb{P}(Y_x = n) = \int_{-1/2}^{1/2} e^{-2\pi i (n - \mu_x) t} \prod_{s \in S_x} \cos(\pi s t) \, \mathrm{d}t. \tag{16}$$

On $|t| \le T/V_x$, the elementary cosine approximation and telescoping products give

$$\left| \prod_s \cos(\pi s t) - e^{-\pi^2 V_x^2 t^2/2} \right| \le C t^4 \sum_s s^4. \tag{17}$$

Since $\sum_s s^4 \le (\rho x)^2 V_x^2$, the integral of this error is

$$O\left(\frac{x^2 T^5}{V_x^3} \right) = o(V_x^{-1}).$$

The Gaussian tail is also $o(V_x^{-1})$. Its full Fourier transform is

$$\int_{\mathbb{R}} e^{-2\pi i (n-\mu_x) t} e^{-\pi^2 V_x^2 t^2/2} \, \mathrm{d}t = \frac{\sqrt{2/\pi}}{V_x} e^{-2(n-\mu_x)^2/V_x^2}.$$

Together with Lemma 4.1, this proves (1). The error is uniform in n because the oscillatory factor has modulus one. On $|n - \mu_x| \le \sigma_x$, the main term is at least $e^{-1/2}/(\sqrt{2\pi}\sigma_x)$, proving positivity. □

Proof of Theorem 1.2. First let $\rho = p/q$ be rational. For large x, Proposition 2.1 gives $m_x \geq 4p^4$, and consequently $V_x \geq 2p^2 x$. Also

$$W_{x+1} \leq W_x + p^2(x+1). \tag{18}$$

Indeed, the entering elements lie in an interval of length $p/q < p$, which contains at most p integers, each at most $p(x+1)$. Departing elements only decrease the sum.

Fix a sufficiently large initial X. For N with $W_X \leq 2N$, choose the largest integer $x \geq X$ satisfying $W_x \leq 2N$. Such an x exists and is finite because $W_x \to \infty$. Maximality and (18) give

$$0 \leq 2N - W_x < W_{x+1} - W_x \leq p^2(x+1) \leq 2p^2 x \leq V_x.$$

Thus $|N - \mu_x| \leq \sigma_x$, and Theorem 1.1 applies. For a real ratio, use a rational $1 < \rho' < \rho$. The divisibility assertion was proved in Section 1. □

5. A quantitative local limit theorem

Theorem 5.1 (Convergence rate)**.** *For all sufficiently large x,*

$$\sup_{n\in\mathbb{Z}} \left| \mathbb{P}(Y_x = n) - \frac{e^{-(n-\mu_x)^2/(2\sigma_x^2)}}{\sqrt{2\pi}\,\sigma_x} \right| \leq \frac{C}{\sigma_x} \frac{(\log L)^{5/2}}{L^2}. \tag{19}$$

Proof. Repeat the proof in Section 4 with $T = \sqrt{2\log L}$. The major-arc approximation (17) contributes at most

$$\frac{C}{V_x} \frac{T^5}{L^2}.$$

The no-wrap tail contributes $O(V_x^{-1} e^{-2T^2}/T)$, and the Gaussian tail contributes $O(V_x^{-1} e^{-\pi^2 T^2/2}/T)$. The two terms in (15), multiplied by V_x, decay faster than any inverse power of L. These four error sources give (19). □

Corollary 5.2 (An expanding Gaussian window)**.** *For each fixed $0 < \varepsilon < 1$ and all sufficiently large x, every integer satisfying*

$$|n - \mu_x| \leq 2\sigma_x \sqrt{(1-\varepsilon)\log L}$$

is a subset sum of S_x.

Proof. Throughout this interval the Gaussian term is at least $(\sqrt{2\pi}\sigma_x)^{-1} L^{-2+2\varepsilon}$. Its ratio to the error bound tends to infinity. □

6. Tilting, linear windows, and representation counts

The rigidity estimate is independent of the Bernoulli parameters. A uniform version that allows them to become small will also settle the interior spectrum.

Theorem 6.1 (Uniform tilted local theorem)**.** *Let $(\xi_s)_{s\in S_x}$ be independent with $\xi_s \sim \mathrm{Bern}(p_s)$, and put*

$$Y = \sum_s s\xi_s, \quad \mu = \sum_s s p_s, \quad v^2 = \sum_s s^2 p_s(1-p_s), \quad b_x = \min_s p_s(1-p_s).$$

There are fixed constants $A, C > 0$ such that, whenever

$$b_x \geq A \frac{\log L}{L}, \tag{20}$$

we have, uniformly in the parameters and in $n \in \mathbb{Z}$,

$$\mathbb{P}(Y = n) = \frac{1}{\sqrt{2\pi}\,v} e^{-(n-\mu)^2/(2v^2)} + O\!\left(\frac{x}{v^2}\right). \tag{21}$$

In particular, the error divided by $1/v$ is $O((L\sqrt{b_x})^{-1}) = o(1)$.

Proof. For a single Bernoulli factor,

$$|1-p+pe^{2\pi ist}|^2 = 1-4p(1-p)\sin^2(\pi st). \tag{22}$$

Since $\sin(\pi u) \geq 2u$ for $0 \leq u \leq 1/2$, the modulus of the product is at most $e^{-8b_x Q_x(t)}$.

Choose δ in Proposition 3.1 small enough for the Taylor estimates below. Off $|t| \leq \delta/x$, layering gives

$$\int |\mathbb{E}e^{2\pi itY}|\,\mathrm{d}t \leq e^{-8b_x z_0 L^2} + \frac{8b_x}{x}(L+2)^C \int_{\kappa L}^{z_0 L^2} e^{-(8b_x - C\log(L+2)/L)z}\,\mathrm{d}z.$$

Choose A large enough that the coefficient of z is at least $4b_x$ and the integral is $O(x^{-1}L^{-4})$. The first term has this bound as well for all large L, since $b_x L^2 \geq AL\log L$.

For the centered factor, uniformly in $0 \leq p \leq 1$ and small real z,

$$\log\left((1-p)e^{-ipz} + pe^{i(1-p)z}\right) = -\tfrac{1}{2}p(1-p)z^2 + O(p(1-p)|z|^3).$$

To verify uniformity, differentiate $\log(1-p+pe^{iz}) - ipz$ three times. The third derivative is bounded by $Cp(1-p)$ when $|z|$ is small, since the denominator is bounded away from zero. Summing with $z = 2\pi st$ gives

$$\mathbb{E}e^{2\pi it(Y-\mu)} = \exp(-2\pi^2 v^2 t^2 + R(t)), \qquad |R(t)| \leq Cxv^2|t|^3.$$

For $|t| \leq \delta/x$, reduce δ so that the real Gaussian decay absorbs this remainder. Then

$$\left|\mathbb{E}e^{2\pi it(Y-\mu)} - e^{-2\pi^2 v^2 t^2}\right| \leq Cxv^2|t|^3 e^{-cv^2t^2}.$$

Its integral is $O(x/v^2)$. The Gaussian tail beyond δ/x is smaller, because $v^2/x^2 \gg b_x L^2$. Fourier inversion proves (21); the minor-arc bound is absorbed since $v^2 \ll x^2 L^2$. □

Corollary 6.2 (Full linear window)**.** *For each fixed $0 < \varepsilon < 1/2$, every integer in*

$$[\varepsilon W_x, (1-\varepsilon)W_x]$$

is a subset sum of S_x for all sufficiently large x.

Proof. Use the common parameter $p_s = \theta = n/W_x \in [\varepsilon, 1-\varepsilon]$. The tilted mean is exactly n, and Theorem 6.1 makes its point probability positive. □

Proposition 6.3 (Many integer scales)**.** *Put $C_W = (\rho-1)/(2\alpha\beta\gamma)$. For fixed $0 < \varepsilon < 1/2$, let*

$$\mathcal{X}_N(\varepsilon) = \{x \in \mathbb{Z}_{\geq 1} : N/(1-\varepsilon) \leq W_x \leq N/\varepsilon\}.$$

Every sufficiently large N is represented by each of these bands, and

$$|\mathcal{X}_N(\varepsilon)| \sim \frac{1}{C_W}\left(\frac{1}{\varepsilon} - \frac{1}{1-\varepsilon}\right)\frac{N}{(\log N)^2}. \tag{23}$$

The count is over integer lower endpoints x; distinct x may give the same band.

Proof. The representation assertion is Corollary 6.2. By (7), $W_x \sim C_W x(\log x)^2$. If $x = uN/(\log N)^2$ with u in a fixed compact subinterval of $(0,\infty)$, then $W_x/N \to C_W u$ uniformly. The coarse bounds first restrict all eligible x to such a compact range. Sandwich the endpoints with an arbitrarily small fixed relative error and count integers. No monotonicity of W_x is needed.

Different x can give the same S_x. In the fixed-factor scale range occurring here, the band changes only when an element crosses an endpoint. Lemma 2.2, or a finite subdivision into fixed-ratio bands, bounds the number of such changes by $O((\log N)^2)$. □

Define the unweighted number of representations by

$$r_x(n) = |\{A \subset S_x : \sum_{s\in A} s = n\}|,$$

and let $h(u) = -u\log u - (1-u)\log(1-u)$ denote binary entropy, with natural logarithms.

Theorem 6.4 (Representation counts at linear scales)**.** *Fix $0 < \varepsilon < 1/2$. For every integer $\varepsilon W_x \le n \le (1-\varepsilon)W_x$, there is a unique real λ satisfying*

$$\sum_{s\in S_x} \frac{s}{1+e^{\lambda s}} = n.$$

Set $p_s = (1+e^{\lambda s})^{-1}$, $v_\lambda^2 = \sum_s s^2 p_s(1-p_s)$, and $Z_x(\lambda) = \prod_s (1+e^{-\lambda s})$. Uniformly in this range,

$$r_x(n) = \frac{1+O(L^{-1})}{\sqrt{2\pi}\, v_\lambda} e^{\lambda n} Z_x(\lambda) \tag{24}$$

$$= \frac{1+O(L^{-1})}{\sqrt{2\pi}\, v_\lambda} \exp\left(\sum_{s\in S_x} h(p_s)\right). \tag{25}$$

Proof. The defining sum is continuous and strictly decreasing from W_x to zero as λ goes from $-\infty$ to $+\infty$. If $\lambda \ge 0$, then

$$\varepsilon \le n/W_x \le (1+e^{\lambda x})^{-1},$$

so $\lambda x \le \log((1-\varepsilon)/\varepsilon)$. The complementary argument applies when $\lambda < 0$. Hence all p_s stay in a fixed compact subinterval of $(0,1)$, and $v_\lambda \asymp xL$.

Under this product measure, a particular subset A has probability $e^{-\lambda\sum_{s\in A} s}/Z_x(\lambda)$. Therefore

$$\mathbb{P}_\lambda(Y = n) = r_x(n) e^{-\lambda n}/Z_x(\lambda).$$

Apply Theorem 6.1 at its mean n. Finally,

$$h(p_s) = \log(1+e^{-\lambda s}) + \lambda s p_s,$$

so $\sum_s h(p_s) = \log Z_x(\lambda) + \lambda n$, proving the second form. □

Remark 6.5 (Normalization and choice of tilt)*.* In (25), the term λn is already included in the entropy sum; adding it again would double-count it. Here v_λ denotes standard deviation. A common Bernoulli parameter proves existence, but except at $\theta = 1/2$ it assigns different weights to representations with different numbers of summands. The exponential tilt above assigns equal probability to all subsets with the same sum and therefore gives the unweighted count.

7. The spectrum of a single band

Write $\mathcal{R}_x = \{n \in \mathbb{Z}_{\ge 0} : r_x(n) > 0\}$. The complement map gives the exact symmetry

$$r_x(n) = r_x(W_x - n), \qquad \mathcal{R}_x = W_x - \mathcal{R}_x. \tag{26}$$

Theorem 7.1 (Interior saturation)**.** *There is a constant $C_0 > 0$ such that, for all sufficiently large x,*

$$[C_0 xL\log L,\ W_x - C_0 xL\log L] \cap \mathbb{Z} \subset \mathcal{R}_x. \tag{27}$$

Proof. For a target n in this interval, use $\theta = n/W_x$. Since $W_x \le C'_W xL^2$ for a fixed constant C'_W, we have

$$\theta(1-\theta) \ge \tfrac{1}{2}\min(\theta, 1-\theta) \ge \frac{C_0}{2C'_W}\frac{\log L}{L}.$$

Choose $C_0 \ge 2AC'_W$. Theorem 6.1 then gives a positive probability at the mean. Its proof integrates Gaussian decay at the tilted variance scale, so no fixed fair-coin cutoff is being used as the tilt degenerates. □

Proposition 7.2 (Sparse bottom edge)**.** *For each fixed $0 < c < 1$, put $y_x = cxL/\log L$. Then*

$$|\mathcal{R}_x \cap [0, y_x]| \le x^{c+o(1)} = o(y_x). \tag{28}$$

Thus the proportion of represented integers in this range tends to zero.

Proof. A subset with sum at most y_x has at most $k_x = \lfloor cL/\log L \rfloor$ elements. Since $m_x \asymp L^2$ and $k_x \le m_x/2$ for large x,

$$|\mathcal{R}_x \cap [0, y_x]| \le \sum_{j=0}^{k_x} \binom{m_x}{j} \le (k_x+1)\left(\frac{em_x}{k_x}\right)^{k_x}.$$

The logarithm of the right side is at most

$$O(\log L) + \frac{cL}{\log L}\big(\log L + \log\log L + O(1)\big) = cL + o(L).$$

This proves the claim. The factor $k_x!$ implicit in the binomial bound is essential to the exponent c. □

Definition 7.3 (Saturation threshold)**.** For all sufficiently large x, define

$$n_*(x) = \min\{k \in \mathbb{Z}_{\ge 0} : k \le W_x/2,\ [k, W_x - k] \cap \mathbb{Z} \subset \mathcal{R}_x\}.$$

The set is nonempty by Theorem 7.1.

Corollary 7.4 (Threshold bounds)**.** *We have*

$$(1-o(1))\frac{xL}{\log L} \le n_*(x) \le C_0 xL\log L + 1. \tag{29}$$

Consequently $n_*(x) = xL^{1+o(1)}$.

Proof. Fix $0 < c < c' < 1$. Proposition 7.2 with c' shows that the number of represented integers up to $c'xL/\log L$ is $o(xL/\log L)$. There is therefore an unrepresented integer h between $cxL/\log L$ and $c'xL/\log L$. For large x, $h < W_x/2$. If an admissible $k \le cxL/\log L$ existed in the definition of $n_*(x)$, then $k \le h \le W_x/2 \le W_x - k$, a contradiction. Thus $n_*(x) > cxL/\log L$. Since this holds for every $c < 1$, it gives the lower bound in (29). Theorem 7.1 gives the upper bound after rounding its lower endpoint upward. Taking logarithms of $n_*(x)/x$ gives $n_*(x) = xL^{1+o(1)}$. □

The unresolved precision is the factor within the $O((\log L)^2)$ gap in (29). For this definition of the threshold, the coarser expression $xL^{1+o(1)}$ is already a conclusion of the proved bounds, rather than an additional conjecture.

8. Rational wells and archimedean harmonics

8.1. **Exact identities and fixed-denominator asymptotics.** Let

$$\widetilde{Q}_x(t) = \sum_{s \in S_x} \sin^2(\pi s t).$$

At the parity frequency,

$$Q_x(1/2) = \tfrac{1}{4}\,|\{s \in S_x : s \text{ is odd}\}|, \qquad \widetilde{Q}_x(1/2) = |\{s \in S_x : s \text{ is odd}\}|. \tag{30}$$

If one base is even, the odd elements form the two-base face omitting that base; their number is asymptotic to a positive constant times L. If all bases are odd, every band element is odd and the depth is instead of order L^2.

Proposition 8.1 (Supported rational centers)**.** *Fix a reduced fraction h/q such that every prime divisor of q divides abc. Then*

$$Q_x(h/q) = O_q(L), \qquad \widetilde{Q}_x(h/q) = O_q(L).$$

Proof. Choose fixed exponent thresholds so that every element whose three exponents exceed their thresholds is divisible by q. Such elements contribute zero. The exceptional elements lie in finitely many coordinate slices, each with $O(L)$ elements by (8). □

For the special case $(a,b,c)=(2,3,5)$ one can be more explicit. For $p\in\{2,3,5\}$ and $k\ge 1$, define

$$M_p(k)=\frac{1}{\varphi(p^k)}\sum_{\substack{u \bmod p^k\\(u,p)=1}}\left\|u/p^k\right\|^2,\qquad \widetilde{M}_p(k)=\frac{1}{\varphi(p^k)}\sum_{\substack{u \bmod p^k\\(u,p)=1}}\sin^2(\pi u/p^k),$$

where φ is Euler's totient function, and put $D_p=\eta/\prod_{r\in\{2,3,5\}\setminus\{p\}}\log r$.

Proposition 8.2 (Fixed-denominator well depths for $(2,3,5)$)**.** *If $q=2^{e_2}3^{e_3}5^{e_5}$ is fixed and $(h,q)=1$, then*

$$Q_x(h/q)=L\sum_{p\in\{2,3,5\}}D_p\sum_{j=0}^{e_p-1}M_p(e_p-j)+o(L), \tag{31}$$

$$\widetilde{Q}_x(h/q)=L\sum_{p\in\{2,3,5\}}D_p\sum_{j=0}^{e_p-1}\widetilde{M}_p(e_p-j)+o(L). \tag{32}$$

The error term depends on q.

Proof. Separate elements according to which exponent is below its threshold e_p. Intersections of two such exceptions contain only $O_q(1)$ band elements: two exponents are fixed, and at most one value of the third is possible. In a single shell with the p-exponent equal to $j<e_p$, the other exponents are at least their thresholds, and cancellation leaves a unit residue modulo p^{e_p-j} multiplied by powers of the other two bases.

Those two bases generate the full unit group. Modulo 3^k and 5^k, the element 2 has orders $2\cdot 3^{k-1}$ and $4\cdot 5^{k-1}$, respectively. This follows by binomial induction from the valuations of 2^2-1 and 2^4-1. Modulo 2^k for $k\ge 3$, powers of 5 give all units congruent to 1 modulo 4, and 3 supplies the other coset; $k=1,2$ are direct. Prescribing the two exponents modulo a common multiple of their orders therefore distributes their products uniformly over the units, by Corollary 2.3. Each shell has $D_pL+o(L)$ elements. Averaging the two test functions proves the formulas. □

At $1/2$ the two energies have exact ratio 4, and at $1/3$ they have exact ratio $27/4$ whenever the denominator is nonzero. At $2/5$, for $(2,3,5)$, their ratio tends to $25/4$; it need not equal that number for a finite band. Moreover,

$$\widetilde{Q}_x(1/3)\sim\frac{3\eta}{4\log 2\log 5}L,\qquad \widetilde{Q}_x(1/9)\sim\frac{5\eta}{4\log 2\log 5}L.$$

Thus a ranking of wells must specify the energy and the regime. In particular, $1/9$ is not asymptotically deeper than $1/3$ in the sine energy. Whether the $o(L)$ error can be sharpened to $O(1)$, as suggested in [H26], is open.

8.2. **Archimedean harmonics.**

Proposition 8.3 (Harmonic identity and its range)**.** *For an integer m, set $t_m=mW_x/V_x^2$. If*

$$|st_m-m|\le\tfrac{1}{2}\quad\text{for every } s\in S_x, \tag{33}$$

then the exact identity

$$Q_x(t_m)=m^2\left(m_x-\frac{W_x^2}{V_x^2}\right) \tag{34}$$

holds. For every fixed integer satisfying

$$|m|<\frac{\rho+1}{2(\rho-1)},$$

condition (33) *holds for all sufficiently large x. In that case*

$$xt_m\longrightarrow\frac{2m}{\rho+1},\qquad \frac{Q_x(t_m)}{m_x}\longrightarrow m^2c_{\mathrm{arch}}(\rho),\quad c_{\mathrm{arch}}(\rho)=1-\frac{2(\rho-1)}{(\rho+1)\log\rho}>0.$$

Proof. Under (33), the nearest-integer vector can be chosen constantly equal to m. Expanding $\sum_s (st_m - m)^2$ proves (34). By (7), $xW_x/V_x^2 \to 2/(\rho+1)$. Thus

$$\max_s |st_m - m| \le |m| \frac{\rho-1}{\rho+1} + o(1),$$

which proves the asserted sufficient condition. The remaining limits follow from the same moment asymptotics. Strict positivity follows from strict Cauchy–Schwarz for the nonconstant function e^u on $[0,\eta]$. □

For $\rho = 3/2$, the fixed nonzero positive harmonics covered by the proposition are $m = 1, 2$. They have depth of order L^2 and therefore eventually leave every fixed εL sublevel set.

8.3. **What a full well classification would require.**

Conjecture 8.4 (Finite-well confinement). For each fixed $E > 0$, there is a finite set $\mathcal{W}_E \subset \mathbb{Q}/\mathbb{Z}$, containing zero and with denominators supported on the primes dividing abc, such that

$$\{t : Q_x(t) \le EL\} \subset \bigcup_{r \in \mathcal{W}_E} \{t : \|t - r\| \le C_E/(x\sqrt{L})\}$$

for all sufficiently large x.

This would imply the improved measure bound $O_E(1/(x\sqrt{L}))$. Propositions 8.1–8.3 establish important examples and identities, but do not classify all low-energy frequencies. Proposition 3.1 alone does not prove Conjecture 8.4.

9. Congruence fluctuations and the sine energy

Let all Bernoulli parameters equal θ, and put $u = \theta(1-\theta)$. The uncentered characteristic function is

$$\Phi_\theta(t) = \prod_{s \in S_x} (1 - \theta + \theta e^{2\pi i s t}).$$

Proposition 9.1 (Exact residue formulas)**.** *For every integer $q \ge 1$ and residue r,*

$$\mathbb{P}_\theta(Y \equiv r \pmod q) = \frac{1}{q} \sum_{h=0}^{q-1} e^{-2\pi i h r/q} \Phi_\theta(h/q). \tag{35}$$

If o_x is the number of odd band elements, then

$$\mathbb{P}_\theta(Y \text{ even}) - \mathbb{P}_\theta(Y \text{ odd}) = (1-2\theta)^{o_x}. \tag{36}$$

For $u \to 0$, uniformly in t,

$$|\Phi_\theta(t)| = \exp\big(-2u\widetilde{Q}_x(t) + O(u^2 \widetilde{Q}_x(t))\big). \tag{37}$$

Proof. Equation (35) is finite Fourier inversion on $\mathbb{Z}/q\mathbb{Z}$. At $t = 1/2$, each even element contributes one and each odd element contributes $1 - 2\theta$, giving (36). Finally, (22) gives

$$\log|\Phi_\theta(t)| = \tfrac{1}{2} \sum_s \log(1 - 4u \sin^2(\pi s t)).$$

Expand $\log(1-z) = -z + O(z^2)$ for small u and use $\sin^4 \le \sin^2$. □

Thus Q_x controls geometric sublevel sets, while $\widetilde{Q}_x$ controls small-tilt amplitudes. The complex product Φ_θ also contains phase information, which its modulus does not determine.

Suppose exactly one base is even and let $D_{\text{odd}} = \eta$ divided by the product of the logarithms of the other two bases. Then $o_x \sim D_{\text{odd}} L$. For the threshold-sized tilt $\theta = A \log L/L$ with fixed $A > 0$, (36) yields

$$\mathbb{P}_\theta(Y \text{ even}) - \mathbb{P}_\theta(Y \text{ odd}) = L^{-2AD_{\text{odd}} + o(1)}. \tag{38}$$

Likewise, fixed-denominator wells with $\widetilde{Q}_x(h/q) \sim DL$ have modulus $L^{-2AD+o(1)}$ at this tilt. These are polynomially small residue biases, not by themselves polynomial ratios between individual even

and odd representation counts. At $\theta = 1/2$, parity is exactly balanced once $o_x > 0$, whereas a nonzero exponential approximation to the parity product would be wrong.

Question 9.2 (Local well corrections). Can one integrate the individual rational neighborhoods, with their phases and local curvatures, to obtain a uniform expansion for individual point probabilities or unweighted representation counts near the saturation threshold?

The exact residue identities do not answer this local question. Such an expansion would need quantitative control of the intermediate Fourier region as well as of the wells themselves. The residue-profile correlations observed numerically in [H26] motivate this question.

10. Controlling the number of summands

Return to fair Bernoulli variables and set $M = \sum_{s\in S_x} \xi_s$. Define

$$g_0 = \gcd(a-1, b-1, c-1), \qquad \Sigma_x = \frac{1}{4}\begin{pmatrix} V_x^2 & W_x \\ W_x & m_x \end{pmatrix}.$$

All band elements are congruent to one modulo g_0, so $Y \equiv M \pmod{g_0}$. The covariance matrix is nondegenerate for large x; Lemma 2.2 gives

$$(t,\phi)\, 4\Sigma_x (t,\phi)^{\mathsf{T}} \asymp L^2(x^2t^2+\phi^2), \qquad \sqrt{\det \Sigma_x} \asymp xL^2. \tag{39}$$

The strict inequality needed here is the same one as $c_{\mathrm{arch}}(\rho) > 0$.

Lemma 10.1 (Affine energy reduction). *Put $\mathcal{E}_x(t,\phi) = \sum_{s\in S_x} \|st+\phi\|^2$. There is an integer $1 \le d_x \le K$, depending on the grid but not on (t,ϕ), such that*

$$\|d_x\phi\|^2 \le C\mathcal{E}_x(t,\phi)/L^2, \qquad Q_x(d_x t) \le C\mathcal{E}_x(t,\phi). \tag{40}$$

Consequently, for $0 \le z \le z_1 L^2$,

$$\operatorname{meas}\{(t,\phi)\in\mathbb{T}^2 : \mathcal{E}_x(t,\phi) \le z\} \le \frac{C\sqrt{z}}{xL}\exp\!\left(C\left(1+\frac{z}{L}\right)\log(L+2)\right). \tag{41}$$

Proof. The grid has $\gg L^2$ edges and bounded degree. In (5), $A_{vw} \ne B_{vw}$ because $s_v \ne s_w$. Some gap $d_x = |B_{vw} - A_{vw}| \le K$ occurs on $\gg L^2$ edges. On each such edge,

$$\|d_x\phi\| \le B_{vw}\,\|s_w t+\phi\| + A_{vw}\,\|s_v t+\phi\|.$$

Square and sum, using bounded degree, to prove the first inequality. For every s,

$$\|d_x st\| \le d_x\,\|st+\phi\| + \|d_x\phi\|.$$

Sum its square and use $m_x \ll L^2$ to obtain the second. The first inequality confines ϕ to a bounded number of intervals of total length $O(\sqrt{z}/L)$; the second and Proposition 3.1 bound the allowable measure of t. Multiplication by d_x preserves Haar measure on the circle. Decrease z_1 to stay within the range of (9). □

Lemma 10.2 (Excluding spurious torsion centers). *Fix an integer $D \ge 1$. Uniformly over $1 \le d \le D$ and residues $j, h \pmod d$, either*

$$(j/d, h/d) \in \mathcal{D} := \{(r/g_0, -r/g_0) : r \in \mathbb{Z}/g_0\mathbb{Z}\} \subset \mathbb{T}^2,$$

or at least $c_D L$ elements of S_x satisfy $js + h \not\equiv 0 \pmod d$, for all sufficiently large x.

Proof. It suffices to test prime-power divisors p^e of d. If p divides one of the bases, say a, take the a-exponent above a fixed threshold; then $s \equiv 0 \pmod{p^e}$. Unless $h \equiv 0$, a positive proportion of the band violates the congruence. If $h \equiv 0$ but $j \not\equiv 0$, all elements on the face with a-exponent zero violate it, since they are units modulo p^e; this face has $\asymp L$ elements.

If p divides none of the bases, prescribe the three exponents modulo $\varphi(p^e)$ to produce residues $s \equiv 1, a, b, c \pmod{p^e}$. Each prescription has $\asymp L^2$ elements by Corollary 2.3. Unless

$$j + h \equiv 0, \qquad j(a-1) \equiv j(b-1) \equiv j(c-1) \equiv 0 \pmod{p^e},$$

one of these families violates the congruence. The conditions for every prime-power divisor are precisely $j + h \equiv 0 \pmod{d}$ and $jg_0 \equiv 0 \pmod{d}$, which describe $\mathcal{D}$. There are only finitely many tests with $d \le D$, so one positive constant suffices. □

Corollary 10.3 (The exact difference lattice)**.** *For all sufficiently large x,*

$$\gcd\{s - s' : s, s' \in S_x\} = g_0.$$

The lattice generated by the vectors $(s, 1)$ is therefore exactly $\{(n, m) \in \mathbb{Z}^2 : n \equiv m \pmod{g_0}\}$, of determinant g_0.

Proof. Let g_x be the gcd of the differences and fix $s_0 \in S_x$. Since $\gcd(S_x) = 1$, we have $(s_0, g_x) = 1$. Reducing an edge equation modulo g_x gives $g_x \mid A_{vw} - B_{vw}$, hence $g_x \le K$. The character $(1/g_x, -s_0/g_x)$ annihilates every $(s, 1)$, so Lemma 10.2 places it in $\mathcal{D}$. Thus $g_x \mid g_0$. The reverse divisibility follows from $s \equiv 1 \pmod{g_0}$. Finally, differences generate $(g_0, 0)$ and one vector $(s_0, 1)$ supplies the second lattice generator. □

Theorem 10.4 (Joint local limit theorem)**.** *For $z = (n - W_x/2, m - m_x/2)^{\mathsf{T}}$, uniformly in $(n, m) \in \mathbb{Z}^2$ with $n \equiv m \pmod{g_0}$,*

$$\mathbb{P}(Y_x = n, M = m) = \frac{g_0}{2\pi\sqrt{\det \Sigma_x}} e^{-z^{\mathsf{T}}\Sigma_x^{-1} z/2} + o((xL^2)^{-1}). \tag{42}$$

Outside this lattice the probability is exactly zero.

Proof. The modulus of the joint characteristic function is at most $e^{-2\mathcal{E}_x(t,\phi)}$. We first locate the region $\mathcal{E}_x < \kappa_1 L$ for a sufficiently small fixed κ_1. Lemma 10.1 and (10) imply

$$t = j/d_x + u, \quad |u| \le \delta/x, \qquad \phi = h/d_x + v, \quad |v| \ll L^{-1/2}.$$

Here δ can be chosen arbitrarily small in advance. If $(j/d_x, h/d_x) \notin \mathcal{D}$, Lemma 10.2 gives at least cL terms whose rational phase has distance at least $1/K$ from an integer. Choose δ so that $|su+v| \le 1/(2K)$ for large x. These terms force $\mathcal{E}_x \ge c'L$, contradicting the choice of κ_1.

Thus the center belongs to $\mathcal{D}$. At these centers all phases are integral, and, reducing δ once more, $|su + v| < 1/2$ for every s. There is no wrap, so

$$\mathcal{E}_x(t, \phi) = V_x^2 u^2 + 2W_x uv + m_x v^2 \asymp L^2(x^2u^2 + v^2). \tag{43}$$

On the complementary region $\mathcal{E}_x \ge \kappa_1 L$, layering with (41) gives an exponentially small bound of the form

$$O(x^{-1}L^C e^{-cL}) + e^{-cL^2} = o((xL^2)^{-1}).$$

Around each point of $\mathcal{D}$, take the box $|u| \le T/(xL)$, $|v| \le T/L$, with $T = L^{1/4}$. Outside these boxes but in the small-energy region, (43) gives a Gaussian tail $o((xL^2)^{-1})$. Inside a box, the centered product is

$$\prod_s \cos(\pi(su + v)).$$

Telescoping the cosine approximation bounds its difference from $\exp(-2\pi^2(u, v)\Sigma_x(u, v)^{\mathsf{T}})$ by $C\sum_s(su + v)^4 \ll T^4/L^2$. Multiplying by the box area gives error $O(T^6/(xL^4)) = o((xL^2)^{-1})$. The Gaussian Fourier transform is the bivariate density in (42) without the factor g_0.

Translation of the uncentered characteristic function by $(r/g_0, -r/g_0)$ leaves it unchanged. The inversion phases from the g_0 boxes sum to $\sum_{r=0}^{g_0-1} e^{-2\pi i r(n-m)/g_0}$, which is g_0 on the stated lattice and zero otherwise. All error estimates are independent of (n, m). □

Corollary 10.5 (Prescribing a central summand count)**.** *Put*

$$\tau_x = \tfrac{1}{2}\sqrt{m_x - W_x^2/V_x^2} \asymp L.$$

For any fixed $B > 0$, all sufficiently large x, and every integer pair satisfying $n \equiv m \pmod{g_0}$ and

$$|n - W_x/2| \le B\sigma_x, \qquad \left|m - \frac{m_x}{2} - \frac{W_x}{V_x^2}(n - W_x/2)\right| \le B\tau_x,$$

there is a subset of S_x with sum n and exactly m elements.

Proof. Completing the square in the Gaussian exponent bounds $z^{\mathsf{T}}\Sigma_x^{-1}z$ by $2B^2$. The main term in (42) then dominates the error. □

If all bases are odd, g_0 is at least two, not necessarily exactly two. For $(5, 9, 13)$ it equals four. The Gaussian approximation is unaffected; only the supporting lattice changes.

11. The boundary of the present energy estimate

Theorem 6.1 reaches $\theta \asymp \log L/L$, corresponding to targets $n \asymp xL\log L$. The present sublevel estimate does not by itself reach $\theta \asymp 1/L$.

Indeed, the competition in the layer integral is between damping $-8\theta(1-\theta)z$ and the exponent $C(1+z/L)\log L$ in (9). At $\theta = A'/L$ and $z = BL\log L$, with fixed positive A', B, their sum is

$$CB(\log L)^2 - 8A'B\log L + O(\log L),$$

which grows. Even a complete classification below a fixed $\varepsilon_0 L$ would leave this intermediate range untreated. Thus the current inequalities do not yield the needed negligible tail at targets of order xL.

The obstruction lies in the present estimates. A sharper intermediate-energy measure bound, or information not captured by absolute values, could remove it. The threshold bounds (29) remain the proved statement.

12. Shrinking bands

Corollary 12.1 (Qualitative shrinking)**.** *There is a function $\varepsilon(N) > 0$ tending to zero such that every sufficiently large integer N is a subset sum of*

$$S \cap [x, (1+\varepsilon(N))x)$$

for some positive integer x.

Proof. For each integer $j \geq 2$, apply Theorem 1.2 with $\rho_j = 1 + 1/j$. Choose increasing thresholds $N_j \geq j$ so that every $N \geq N_j$ is represented with this ratio. For $N \geq N_2$, put $j(N) = \max\{j : N_j \leq N\}$ and $\varepsilon(N) = 1/j(N)$. Then $j(N) \to \infty$ and the conclusion follows. □

This diagonal argument supplies no rate. Hou [H26] proposes ratios of the form $1 + c(\log\log N)^{-1/\mu}$ from quantitative irrationality bounds, and a stronger $1 + c/\log\log N$ in the $(2, 3, 5)$ case. These rates are not established by the fixed-band estimates proved above.

A quantitative proof must control, as $\eta \downarrow 0$, the rounding depth, the exponent-step bound D, the multiplier bound K, the rigidity constants, the moment estimates, and the final sweep. It must fix the ratio as a function of the target N before choosing the lower endpoint x. A finite set of continued fractions does not alone give uniform control at arbitrarily small widths. The rates proposed in [H26], together with the compatibility condition $L \geq \eta^{-3}$, remain conjectural.

13. The rank threshold and non-coprime obstructions

The narrow-band obstruction has a classical precursor: Erdős and Lewin [EL96, p. 838] use the same subset-counting principle for 2, 3-smooth summands in a band of ratio two. The next theorem gives a uniform formulation for two coprime bases and counts targets across all allowable band locations. It concerns narrow bands only, and is consistent with the classical $(2, 3)$ antichain representation [EL96] and with the ratio-six theorem of van Doorn and Everts [DE25].

Theorem 13.1 (Two bases do not suffice)**.** *Let $a, b \geq 2$ be coprime, fix $1 < \rho < \min(a, b)$, and set $S^{(2)} = \{a^k b^\ell : k, \ell \geq 0\}$. Put*

$$\vartheta = \frac{\log\rho\,\log 2}{\log a\log b} < 1.$$

The number of integers $1 \leq N \leq X$ representable as a subset sum of $S^{(2)} \cap [x, \rho x)$ for some positive integer x is at most $X^{\vartheta+o(1)}$. In particular, the band conclusion of Theorem 1.2 fails in rank two.

Proof. By (8), a band has

$$|S^{(2)} \cap [x, \rho x)| = \frac{\log \rho}{\log a \log b} \log x + o(\log x)$$

elements. Uniformly over $1 \le x \le X$, it therefore has at most $X^{\vartheta+o(1)}$ subsets. As x varies in this range, the band changes only when one of its endpoints passes a semigroup element. There are $O((\log X)^2)$ such elements below ρX, and hence only that many distinct bands. A positive sum $N \le X$ cannot require $x > X$. Multiplying the two bounds proves the claim. Finally,

$$\vartheta < \frac{\log 2}{\max(\log a, \log b)} < 1,$$

since coprime bases at least two are distinct and their maximum is at least three. □

Theorem 13.2 (Higher rank)**.** *Let $a_1, \dots, a_r \ge 2$ be pairwise coprime with $r \ge 3$, and fix rational $1 < \rho < \min_i a_i$. For the corresponding full rank-r band, fair Bernoulli subset sums satisfy the local limit theorem* (1)*, with its own mean and variance. Moreover, for every fixed real such ρ, every sufficiently large integer is a subset sum of an antichain in one band.*

Proof. For the full band, $m_x \asymp L^{r-1}$ and $V_x \asymp xL^{(r-1)/2}$. The upper count follows by fixing $r-1$ exponents. For the lower count, choose the last $r-3$ exponents in boxes of side a sufficiently small multiple of L; the residual lower endpoint for the first three bases is at least $\sqrt{x}$, and Proposition 2.1 supplies $\gg L^2$ elements in each of $\gg L^{r-3}$ disjoint slices.

On $|t| \le 2\delta/x$, all full-band factors have Gaussian damping with the full variance. Outside this interval, retain only the rank-three subband on the first three bases. Bound (15) is exponentially small relative to $1/(xL^B)$ for every fixed B, so it is also negligible at the full variance scale. With $T = \sqrt{r \log L}$, the major-arc error is

$$O\left(\frac{1}{V_x} \frac{T^5}{L^{r-1}}\right) = o(V_x^{-1}).$$

This proves the local theorem. The antichain representation already follows by applying Theorem 1.2 to any three bases and taking all other exponents zero. □

Proposition 13.3 (A non-coprime congruence obstruction)**.** *For $S = \{6^k 10^\ell 15^m : k, \ell, m \ge 0\}$, no integer congruent to two modulo three is a sum of a divisibility antichain in S. In particular, it cannot be a subset sum of any band of ratio less than six.*

Proof. The elements not divisible by three are precisely 10^ℓ. They form a divisibility chain, so an antichain contains at most one of them. Each is congruent to one modulo three. All other summands are zero modulo three, leaving only residues zero and one. In a band of ratio less than six, two powers of ten cannot occur together, so the same argument applies directly. □

For this non-coprime example the band gcd can also vary with scale. Direct integer enumeration at $\rho = 3/2$ gives

x	10^4	10^5	10^6
$\lvert S_x \rvert$	3	5	5
$\gcd(S_x)$	20	20	200

These finite values agree with those reported in [H26]. A prospective non-coprime local theorem would need more than a scale-dependent gcd: at these scales the gcd is coprime to three, but the additional residue obstruction still excludes integers. The exceptional residue structure would also have to be incorporated.

14. Effective thresholds and computational status

The fixed-width rounding argument can be supported by a finite certificate: a finite list of rotation points, together with interval bounds on their gaps, suffices to certify the required net. For logarithms of fixed integers, such a certificate can be checked with rigorous enclosures. The rest of an explicit threshold calculation must then propagate the resulting constants through the grid, the code count, the Fourier tails, positivity, and the final sweep.

Hou [H26] proposes an explicit threshold

$$N_0 = \exp(\exp(19387))$$

using the approximations $3^{12} \approx 2^{19}$, $2^{65} \approx 5^{28}$, and $3^{22} \approx 5^{15}$. The note does not provide the complete constant ledger or a finite covering certificate for a specified band ratio. We do not certify that value here. Near-equalities between powers are useful inputs, but they do not by themselves verify the maximum gaps in each rotation net or the downstream analytic constants.

The coverage experiments, harmonic transition scale, residue correlations, and threshold-tilt constants reported in [H26] are likewise numerical observations; no theorem above depends on them. The finite gcd table in Section 13 is independently reproducible by generating integer powers below ρx, retaining products in the band, and taking their gcd.

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