ON THE FIRST OCCURRENCE OF AN INTEGER AS A PREFIX SUM OF DIVISORS

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ABSTRACT. For a positive integer N, let $f(N)$ be the least positive integer whose increasing list of divisors has an initial segment with sum N, when such an integer exists. We prove that f is well-defined for all positive integers $N \neq 2, 5$. Next, answering a question of Erdős, we show that $f(N) = o(N)$ fails in a strong sense, whereas $\limsup_{N\to\infty} f(N)/N = \infty$. Arbitrarily small and arbitrarily large ratios $f(N)/N$ each occur on sets of positive lower density. Moreover, we obtain a doubly exponential upper bound for the lower tail and quantitative lower bounds for the upper tail.

CONTENTS

1. Introduction 2
1.1. Quantitative estimates for tails 2
1.2. Hypothetical limiting distribution 4
1.3. Proof outlines 4
1.4. Notation 5
2. Preliminaries and basic estimates 6
3. Representability 9
3.1. Computational verification for finitely many integers 10
3.2. An explicit representation by three distinct primes 12
4. Lower tails 13
4.1. Upper bounds for the lower tail 13
4.2. Apparent absence of small ratios 15
4.3. A family of integers with large divisor sums 16
4.4. Divisibility properties and the largest smooth divisor 17
4.5. The second-moment estimate 20
5. Upper tails 24
5.1. Sums over large divisors of a least common multiple 24
5.2. Congruence conditions for bounded cofactors 25
5.3. Fixed thresholds 26
5.4. Growing thresholds 27
6. Bounded representations 28
6.1. A first moment in every residue class 29
6.2. Ranges for prime cofactors 31
6.3. Bounded cofactors 31
7. Counting measures and their convergence 34
7.1. Singularity and continuity of subsequential limits 35
7.2. Multiple representations and convergence 37
8. Further questions 39

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References 41

1. Introduction

Write the positive divisors of an integer $n \geqslant 1$ as

$$1 = d_1(n) < d_2(n) < \cdots < d_{\tau(n)}(n) = n.$$

Here $\tau(n)$ is the number of positive divisors of n, and $\sigma(n) := \sum_{d|n} d$ denotes their sum. A positive integer N is *represented by a prefix of divisors* (or simply *represented*) if

$$N = d_1(n) + \cdots + d_k(n) \quad \text{for some } n \in \mathbb{N} \text{ and } 1 \leqslant k \leqslant \tau(n).$$

Let $\mathcal{R}$ be the set of represented positive integers, and define $f\colon \mathcal{R} \to \mathbb{N}$ by

$$f(N) := \min\{n \in \mathbb{N} : N = d_1(n) + \cdots + d_k(n) \text{ for some } 1 \leqslant k \leqslant \tau(n)\}$$

for every $N \in \mathcal{R}$.

The obvious inequality $f(\sigma(n)) \leqslant n$, together with the well-known unboundedness of $\sigma(n)/n$, gives $\liminf_{N\to\infty,\, N\in\mathcal{R}} f(N)/N = 0$; see a comment under [8]. Erdős asked whether $f(N) = o(N)$ holds as $N \to \infty$, whether this might at least hold for almost all N, and whether $\limsup f(N)/N = \infty$. This problem appears in Guy's collection of open problems in number theory [10, Problem B2] and is catalogued as Problem 1054 on Bloom's website *Erdős problems* [2]. Its variant is also mentioned in Erdős and Graham's problem book [7, p. 93]; the convention there omits the divisor 1, which only results in an immaterial shift of the function f. The computed values of f are recorded in OEIS sequence A167485 [8], where it is also asked whether

$$\mathcal{R} = \mathbb{N} \setminus \{2, 5\}. \tag{1.1}$$

A related concept is that of the *aliquot numbers* N, representable as

$$N = d_1(n) + \cdots + d_{\tau(n)-1}(n) \quad \text{for some } n \in \mathbb{N},$$

while the remaining positive integers are known as *untouchable numbers*; see, e.g., [3].

These questions are not easy to answer because the prefix sums of divisors $d_1(n)+\cdots+d_k(n)$ behave quite erratically. Nevertheless, we successfully address all of them by providing nontrivial information on the distribution of the ratios $f(N)/N$; see Theorems 1.1–1.4. Moreover, we give a computer-assisted proof of the property (1.1); see Theorem 3.2.

1.1. Quantitative estimates for tails.

Let $\#S$ denote the cardinality of a finite set S. For $S \subseteq \mathbb{N}$, write

$$\underline{\mathrm{d}}(S) := \liminf_{X\to\infty} \frac{\#(S \cap [1, X])}{X}, \quad \overline{\mathrm{d}}(S) := \limsup_{X\to\infty} \frac{\#(S \cap [1, X])}{X}$$

respectively for *lower* and *upper densities* of S. When these densities agree, their common value is denoted by $\mathrm{d}(S)$. Throughout, "log" denotes the natural logarithm, and logarithmic expressions are used only where they are defined.

The first result bounds the frequency of small ratios $f(N)/N$ uniformly in the cutoff.

Theorem 1.1. *There is an absolute constant $c > 0$ such that, for all sufficiently small $\delta > 0$ and every $X \geqslant 1$,*

$$\#\{N \leqslant X : N \in \mathcal{R},\ f(N) \leqslant \delta N\} \ll X \exp\Big(-\exp((1/\delta)^c)\Big).$$

In particular, for every fixed $M > 0$ this counting function is $O_M(\delta^M X)$, uniformly in X.

An immediate asymptotic consequence of Theorem 1.1 is

$$\overline{\mathrm{d}}\big(\{N \in \mathcal{R} : f(N) \leqslant \delta N\}\big) \ll \exp\Big(-\exp((1/\delta)^c)\Big)$$

for all sufficiently small $\delta > 0$. This fact already strongly refutes Erdős's intuition that $f(N)$ is typically much smaller than N.

Next, arbitrarily small ratios occur on sets of positive lower density.

Theorem 1.2. *For every $\delta > 0$ there is $c_\delta > 0$ such that*

$$\#\{N \leqslant X : N \in \mathcal{R},\ f(N) \leqslant \delta N\} \geqslant c_\delta X$$

for all sufficiently large X.

This time, the asymptotic variant of Theorem 1.2 reads

$$\underline{\mathrm{d}}\big(\{N \in \mathcal{R} : f(N) \leqslant \delta N\}\big) > 0$$

for every $\delta > 0$.

Regarding large ratios $f(N)/N$, our estimate decays more slowly than every negative power of the threshold.

Theorem 1.3. *As $T \to \infty$,*

$$\underline{\mathrm{d}}(\{N \in \mathcal{R} : N \text{ squarefree},\ f(N) > TN\}) \geqslant \left(\frac{e^{-\gamma}}{2} - o(1)\right) \frac{\log\log\log T}{(\log T)(\log\log T)}, \tag{1.2}$$

where γ is Euler's constant. Consequently, $\{N \in \mathcal{R} : f(N) > TN\}$ has positive lower density for every fixed $T > 0$, and

$$\limsup_{\substack{N\to\infty \\ N\in\mathcal{R}}} \frac{f(N)}{N} = \infty.$$

That way Theorem 1.3 also settles the last part of Erdős's question in the affirmative. Moreover, the fact that the set $\{N \in \mathcal{R} : f(N) > TN\}$ has positive lower density for every fixed $T > 0$ is in an even sharper contrast with Erdős's conjecture $f(N) = o(N)$. It remains open whether the upper density of $\{N \in \mathcal{R} : f(N) > TN\}$ tends to zero as $T \to \infty$. Section 6 shows that this condition is equivalent to tightness of the empirical measures and to the assertion that the lower density of the set of integers represented with cofactor at most E tends to 1 as $E \to \infty$.

The final theorem of this type allows the threshold to grow alongside X, as opposed to fixing δ or T as before.

Theorem 1.4. *For every fixed $\eta > 0$ there is $c_\eta > 0$ such that, for all sufficiently large X,*

$$\#\left\{\frac{X}{2} < N \leqslant X : N \in \mathcal{R},\ N \text{ squarefree},\ \frac{f(N)}{N} > \exp\left(\left(\frac{1}{\sqrt{2}} - \eta\right)\sqrt{(\log\log\log N)(\log\log\log\log N)}\right)\right\} \geqslant c_\eta \frac{X}{\log\log\log X}. \tag{1.3}$$

For every fixed $s > 0$ and $\eta > 0$, we have

$$\frac{1}{X}\sum_{\substack{N\leqslant X \\ N\in\mathcal{R}}} \left(\frac{f(N)}{N}\right)^s \geqslant \exp\left(\left(\frac{s}{\sqrt{2}} - \eta\right)\sqrt{(\log\log\log X)(\log\log\log\log X)}\right) \tag{1.4}$$

for all sufficiently large X.

1.2. **Hypothetical limiting distribution.** The tail estimates from Theorems 1.1–1.3 suggest studying the empirical probability measures

$$\nu_X := \frac{1}{R(X)} \sum_{\substack{N \leqslant X \\ N \in \mathcal{R}}} \delta_{f(N)/N}, \tag{1.5}$$

where

$$R(X) := \#(\mathcal{R} \cap [1, X])$$

for $X \geqslant 1$ and δ_x is unit mass at $x \in (0, \infty)$. Since we will see that $R(X) = \lfloor X \rfloor - 2$ for $X \geqslant 5$, the normalization by $R(X)$ is asymptotically equivalent to the one by X. Investigation of weak convergence of the measures (1.5) can be motivated by the classical analogous results for the sequences $\varphi(N)/N$, see [17], or $\sigma(N)/N$, see [4]. We find it plausible that the limiting distribution also exists in the case of the sequence $f(N)/N$, but this might be out of reach of the currently available techniques. Still, the present paper attempts to provide some additional partial results and supporting evidence.

For a factor d of a number $n = ed$, the integer e is called its *cofactor*. Define

$$F_e(d) := \sum_{\substack{r \mid ed \\ r \leqslant d}} r, \quad G_E := \bigcup_{1 \leqslant e \leqslant E} F_e(\mathbb{N}).$$

We prove that the family of probability measures $\{\nu_X : X \geqslant 1\}$ is tight on $[0, \infty)$ if and only if

$$\lim_{E \to \infty} \underline{\mathrm{d}}(G_E) = 1,$$

but this assertion remains a conjecture.

Our distributional conclusions go beyond the tail bounds discussed before. There is one fixed σ-compact Lebesgue-null set $\mathcal{Z} \subset (0, \infty)$ carrying the finite part of every weak subsequential limit of ν_X on $[0, \infty]$. Every such limit has no finite atoms, has 0 in its support, and has infinite positive moments and infinite logarithmic mean. In particular, any subsequential probability limit on $[0, \infty)$ is singular continuous. These conclusions follow from an explicit limiting measure counting all representations and the singularity of the distribution of $\sigma(n)/n$.

We also identify the remaining difficulty for the problem of weak convergence of (1.5). For $t > 0$, let $r_t^*(N)$ count representations $N = F_e(d)$ with $e \geqslant 2$ and $ed \leqslant tN$. Its normalized first moment has a continuous limit $W^*(t)$, and

$$\nu_X([0, t]) = W^*(t) - \frac{1}{X} \sum_{N \leqslant X} (r_t^*(N) - 1)_+ + o(1).$$

Thus, existence of $\lim_{X \to \infty} \nu_X([0, t])$ is equivalent to convergence of the average in the second term of the above sum. This average has lower limit larger than $1/9$ for every $t \geqslant 2$. Consequently it cannot be discarded and it causes serious complications.

1.3. **Proof outlines.** For small ratios $f(N)/N$, the main inspiration for the proofs is Tao's unpublished comment under [2]. It bounded cumulative truncated sums of divisors, and implied a weaker variant of Theorem 1.1, with $\delta^2 X$ on the right-hand side. This already disproved $f(N) = o(N)$. Following his suggestion, we prove an estimate for higher moments of all prefix sums, which eventually gives Theorem 1.1. An intermediate result, with $O_M(\delta^M X)$ for every fixed $M > 0$, was independently obtained by Price [2]. To obtain positive density, we use sums of proper divisors of integers for which $\sigma(n)/n$ is large. The proof adapts the partition according to the largest smooth divisor and the sieve estimates of Luca and Pomerance [12], with Pollack's bounds for ratios of successive aliquot sums [14]. A fixed divisor D with sufficiently large $\sigma(D)/D$ makes the represented ratio arbitrarily small, while the second moment estimate ensures that a positive proportion of the images are distinct.

For large ratios $f(N)/N$, the cofactor separates the problem into a moment bound for large e and an arithmetic obstruction for bounded e. When every prime up to a chosen bound divides $\sigma(n)$ and none divides N, then none divides $\sigma(n) - N$ either. The congruence satisfied by $F_e(d)$ then forces the least common multiple of the divisors of n below e to be large. A Rankin estimate for the resulting sum over large divisors yields Theorem 1.3. A uniform estimate for the number of integers n for which a prescribed prime does not divide $\sigma(n)$, together with a different choice of parameters, gives Theorem 1.4. We use the same parametrization by the cofactor in these estimates and in the construction of measures counting representations.

For the limiting distribution, we fix a cofactor and a residue class modulo $\operatorname{lcm}(1,\dots,e)$. The normalized prefix sum $F_e(n/e)/n$ is then $\sigma(n)/n$ minus a fixed rational number. Davenport's theorem in arithmetic progressions [15] and the large-cofactor moment bound give a limiting measure counting all representations. On $(0,\infty)$, this measure dominates every weak subsequential limit of ν_X. Erdős's singularity theorem [6] consequently implies singularity of the restriction of each such limit to $[0,\infty)$. This argument establishes properties of possible limiting distributions without assuming either tightness or uniqueness.

1.4. **Notation.** For functions F and G, the notations $F \ll G$ and $F = O(G)$ mean that $|F| \leqslant C|G|$ for some constant $C > 0$ throughout the stated range; $F \gg G$ means $G \ll F$, and $F \asymp G$ means that both $F \ll G$ and $G \ll F$ hold. All implied constants are absolute unless their dependence is indicated, for example by a subscript. We write $F = o(G)$ if $F/G \to 0$ in the limit under consideration.

We write $s(n) := \sigma(n) - n$ for the sum of the proper divisors of n (the so-called *aliquot sum*), φ for Euler's totient function, ζ for the Riemann zeta function, and γ for Euler's constant. For integers $a_1,\dots,a_k$, we write $\operatorname{lcm}(a_1,\dots,a_k)$ for their least common multiple and $\gcd(a,b)$ for the greatest common divisor of a and b. We write $v_p(n)$ for the exponent of a prime p in the prime factorization of n, and $p^a \| n$ for exact divisibility. The largest and least prime factors of n are denoted $P^+(n)$ and $P^-(n)$, with $P^+(1) := 1$ and $P^-(1) := \infty$. An integer n is *y-smooth* if $P^+(n) \leqslant y$, and *y-rough* if $P^-(n) > y$. For real $y \geqslant 1$, we use the *primorial* notation

$$y\# := \prod_{\substack{p \leqslant y \\ p \text{ prime}}} p,$$

where each prime not exceeding y occurs exactly once. The empty product is 1; in particular, $y\# = 1$ for $1 \leqslant y < 2$. For real $y, u \geqslant 1$, set

$$\Delta(y) := \frac{\varphi(y\#)}{y\#} = \prod_{\substack{p \leqslant y \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right), \quad \Lambda(u) := \operatorname{lcm}(1,\dots,\lfloor u \rfloor).$$

Thus $\Delta(y)$ is the natural density of integers coprime to $y\#$, and $\Lambda(u)$ is the least common multiple of all positive integers not exceeding u. The symbols $\pi(x)$ and $\pi(x;q,a)$ count primes at most x and primes at most x congruent to a modulo q, respectively, with $\gcd(a,q) = 1$ in the latter notation. We also set $\psi(x) := \sum_{p^a \leqslant x} \log p$, the so-called *second Chebyshev function*. We use $u_+ := \max(u, 0)$ for the positive part of a real number. Finally, for integers $j \geqslant 0$ and $n \geqslant 1$, we write

$$\sigma_j(n) := \begin{cases} d_1(n) + \cdots + d_{\tau(n)-j}(n), & 0 \leqslant j < \tau(n), \\ 0, & j \geqslant \tau(n) \end{cases}$$

for the *prefix sum* of all but j largest divisors of n. We interpret σ_0 as σ.

2. Preliminaries and basic estimates

Using the notation introduced in Section 1, define

$$g_e(n) := \sum_{\substack{r\mid n\\ r\geqslant e}} \frac{1}{r}.$$

Since the final divisor occurs in its own prefix,

$$F_e(d) \geqslant d, \quad F_e(d) \leqslant X \implies ed \leqslant eX.$$

Mertens' theorem and Chebyshev's estimates give

$$\Delta(y) \sim \frac{e^{-\gamma}}{\log y}, \quad \log(y\#) \ll y, \quad \log \Lambda(u) = \psi(u) \leqslant C_\Lambda u,$$

where $C_\Lambda > 0$ is absolute; see, for example, [18].

The involution $r \mapsto ed/r$ gives the following identity for the prefix sum.

Lemma 2.1. *For all positive integers d, e,*

$$F_e(d) = ed\, g_e(ed). \tag{2.1}$$

Proof. The involution $r \mapsto ed/r$ exchanges the divisors $r \geqslant e$ of ed with its divisors at most d. □

The omitted divisors force a congruence whose modulus depends on only finitely much divisibility information. Fix $d \geqslant 1$ and $e \geqslant 2$, put $n := ed$, and let

$$D := \{j : 1 \leqslant j < e,\ j \mid n\}$$

be the set of divisors of n below e. Put

$$M := \operatorname{lcm} D, \quad C := \sum_{j\in D} \frac{M}{j}, \quad g := \gcd(e, M), \quad K_{e,d} := \frac{e}{g}\, C.$$

Since $1 \in D$, these are positive integers. In fact $K_{e,d} \geqslant 2$: if $e/g > 1$ this is immediate, while $e/g = 1$ implies $C \geqslant M \geqslant e \geqslant 2$. For $1 \leqslant j < e$ write $h := \gcd(j, e)$. Then

$$j \mid ed \quad\Longleftrightarrow\quad \frac{j}{h} \mid \frac{e}{h}\, d \quad\Longleftrightarrow\quad \frac{j}{h} \mid d,$$

because j/h and e/h are coprime. Thus D depends only on the residue of d modulo

$$\operatorname{lcm}\big(\{j/\gcd(j, e) : 1 \leqslant j < e\}\big).$$

In particular, for fixed e the integer $K_{e,d}$ takes only finitely many values.

Lemma 2.2. *With the notation above,*

$$F_e(d) = \sigma(n) - \frac{n}{M}\, C, \tag{2.2}$$

the integer M/g divides d, and

$$F_e(d) \equiv \sigma(n) \pmod{K_{e,d}}. \tag{2.3}$$

Moreover $K_{e,d}$ is the largest modulus for which (2.3) *holds for every d with the same set D.*

Proof. The map $r \mapsto n/r$ is an involution of the divisors of n, and $r > d$ exactly when $n/r < e$. Hence

$$\sigma(n) - F_e(d) = \sum_{\substack{r\mid n\\ r>d}} r = \sum_{j\in D} \frac{n}{j} = \frac{n}{M} \sum_{j\in D} \frac{M}{j} = \frac{n}{M}\, C.$$

Every $j \in D$ divides n, so $M \mid n$ and n/M is an integer. This proves (2.2). Since $g = \gcd(e, M)$, the integers M/g and e/g are coprime. From $M \mid ed$ we get $(M/g) \mid (e/g)d$, hence $(M/g) \mid d$. Therefore

$$\frac{n}{M} = \frac{ed}{M} = \frac{e}{g} \cdot \frac{d}{M/g}$$

is e/g times an integer, and by (2.2) the difference $\sigma(n) - F_e(d)$ is a multiple of $(e/g)C = K_{e,d}$. For the last assertion take $d_0 = M/g$, so that $ed_0 = \operatorname{lcm}(e, M)$. Every $j \in D$ divides M, hence ed_0. Conversely ed_0 divides ed, so every divisor of ed_0 below e lies in D. Thus d_0 has the same set D, and at d_0 the difference $(ed_0/M)C$ equals $(e/g)C$ exactly. Any modulus valid for every d with this set D divides that difference, hence divides $K_{e,d}$. □

If $P^-(d) \geqslant e$, then D is the set of proper divisors of e. Indeed, for $j < e$ the integer $j/\gcd(j, e)$ is below $e \leqslant P^-(d)$, so it divides d only if it equals 1, that is, only if $j \mid e$. In this case $M \mid e$, $g = M$, and

$$K_{e,d} = \sum_{\substack{j \mid e \\ j < e}} \frac{e}{j} = \sigma(e) - 1.$$

So on e-rough inputs, $F_e(d) \equiv \sigma(ed) \pmod{\sigma(e) - 1}$.

The sum of divisors is divisible by any fixed positive integer for almost all numbers. The following lemma and the one after are somewhat folklore, but we include them for completeness.

Lemma 2.3. *For every fixed positive integer V,*

$$\#\{n \leqslant Y : V \nmid \sigma(n)\} = o_V(Y)$$

as $Y \to \infty$.

Proof. It suffices to prove the assertion when $V = q$ is a prime power. For every prime $r \equiv -1 \pmod{q}$, the condition $r \| n$ implies

$$q \mid r + 1 = \sigma(r) \mid \sigma(n).$$

Therefore, if $q \nmid \sigma(n)$, none of these primes divides n exactly once. For any finite set $\mathcal{P}$ of such primes, the Chinese remainder theorem gives

$$\limsup_{Y \to \infty} \frac{1}{Y} \#\{n \leqslant Y : q \nmid \sigma(n)\} \leqslant \prod_{r \in \mathcal{P}} \left(1 - \frac{1}{r} + \frac{1}{r^2}\right).$$

The reciprocal sum of the primes in the reduced residue class $-1 \pmod{q}$ diverges. Letting $\mathcal{P}$ increase through this class makes the displayed product tend to zero. Factoring a general fixed V into its prime-power divisors and applying the union bound completes the proof. □

This divisibility property forces the range of the full divisor sum to have density zero.

Lemma 2.4. *The set $\sigma(\mathbb{N}) = \{\sigma(n) : n \geqslant 1\}$ has asymptotic density zero.*

Proof. If $N = \sigma(n) \leqslant X$, then $n \leqslant X$. For every fixed integer $q \geqslant 2$, the values divisible by q number at most X/q, while all remaining values arise from $o_q(X)$ exceptional integers n by Lemma 2.3. Hence

$$\overline{\mathrm{d}}(\sigma(\mathbb{N})) \leqslant \frac{1}{q}.$$

Letting $q \to \infty$ proves the claim. □

For a prime q and a real number $y \geqslant 1$, define

$$B_q(y) := \#\{n \leqslant y : q \nmid \sigma(n)\}.$$

The estimate below for odd primes is weaker than the bound $B_q(y) \ll y/(\log y)^{1/(q-1)}$, valid for $y \geqslant 3$ uniformly over primes q, given in [14, Lemma 2.1] and attributed there to Pomerance.

We include a proof of the weaker form needed here using a mean-value theorem for nonnegative multiplicative functions and the Siegel–Walfisz theorem [18].

Lemma 2.5. *There are absolute constants $c_0, c_1 > 0$ such that, for all sufficiently large y and every prime*

$$3 \leqslant q \leqslant c_0 \frac{\log\log y}{\log\log\log y},$$

one has

$$B_q(y) \ll y \exp\left(-c_1 \frac{\log\log y}{q}\right).$$

Also $B_2(y) \ll \sqrt{y}$.

Proof. For prime powers define the multiplicative function h_q by

$$h_q(p^\nu) := \begin{cases} 0, & p \equiv -1 \pmod q \text{ and } \nu = 1, \\ 1, & \text{otherwise.} \end{cases}$$

If $p \equiv -1 \pmod q$ and $v_p(n) = 1$, then $q \mid \sigma(p) \mid \sigma(n)$. Hence $\mathbb{1}_{q\nmid\sigma(n)} \leqslant h_q(n)$. Since $0 \leqslant h_q(p^\nu) \leqslant 1$, the Halberstam–Richert mean-value bound, in the form [14, Lemma 2.4], applies with $\lambda_1 = \lambda_2 = 1$ in the notation of that lemma. It gives, with an absolute implied constant,

$$B_q(y) \leqslant \sum_{n\leqslant y} h_q(n) \ll y \exp\left(\sum_{p\leqslant y} \frac{h_q(p)-1}{p}\right) = y \exp\Bigg(-\sum_{\substack{p\leqslant y \\ p\equiv -1 \pmod q}} \frac{1}{p}\Bigg).$$

Choose $z = \exp(Cq^2)$ with C sufficiently large. After decreasing c_0 if necessary, the stated range for q ensures that $z \leqslant y$ for all sufficiently large y. By the Siegel–Walfisz theorem [18], uniformly for $t \geqslant z$,

$$\pi(2t; q, -1) - \pi(t; q, -1) \gg \frac{t}{q\log t}.$$

Summing over dyadic intervals from z to y therefore gives

$$\sum_{\substack{z<p\leqslant y \\ p\equiv -1 \pmod q}} \frac{1}{p} \gg \frac{1}{q}\log\frac{\log y}{\log z} = \frac{1}{q}\big(\log\log y - 2\log q + O(1)\big).$$

In the stated range the last parenthesis is at least $\frac{1}{2}\log\log y$ for large y, proving the first assertion. Finally, $\sigma(n)$ is odd exactly when n is a square or twice a square, so $B_2(y) \ll \sqrt{y}$. □

The moment estimate in the next lemma extends the argument of Tao [2], which could be viewed as the first moment approach, while we intend to bound higher moments. The same can be said about the slightly more technical Lemma 4.1 in a later section. Keeping the condition $e \mid n$ in the moment expansion gives the decay E^{1-k} needed below.

Lemma 2.6. *For every integer $k \geqslant 2$ there is a constant C_k satisfying*

$$C_k \leqslant (2k+2)^{2^{k+1}} \tag{2.4}$$

such that, for all $E, Z \geqslant 1$,

$$\sum_{n\leqslant Z} \sum_{\substack{e\mid n \\ e>E}} g_e(n)^k \leqslant C_k Z E^{1-k}. \tag{2.5}$$

Consequently, for all real $A, X \geqslant 1$, the number of integers $N \leqslant X$ for which there are $e, d \in \mathbb{N}$ satisfying $N = F_e(d)$, $e > E$, and $ed \leqslant AN$ is at most

$$C_k A^{k+1} E^{1-k} X. \tag{2.6}$$

Proof. Expanding the kth power and counting common multiples gives

$$\sum_{n\leqslant Z}\sum_{\substack{e\mid n\\ e>E}} g_e(n)^k = \sum_{e>E}\sum_{r_1,\dots,r_k\geqslant e}\frac{1}{r_1\cdots r_k}\left\lfloor\frac{Z}{\operatorname{lcm}(e,r_1,\dots,r_k)}\right\rfloor$$
$$\leqslant Z\sum_{e>E}\sum_{r_1,\dots,r_k\geqslant e}\frac{1}{r_1\cdots r_k\,\operatorname{lcm}(e,r_1,\dots,r_k)}.$$

Set $x_0=e$, $x_i=r_i$, and $R=x_0\cdots x_k$. Since $r_1\cdots r_k\geqslant e^k$ and $e>E$,

$$\frac{R^{1/(k+1)}}{r_1\cdots r_k}\leqslant e^{1-k}\leqslant E^{1-k}.$$

Consequently,

$$\frac{1}{r_1\cdots r_k\,\operatorname{lcm}(e,r_1,\dots,r_k)}\leqslant E^{1-k}\frac{R^{-1/(k+1)}}{\operatorname{lcm}(x_0,\dots,x_k)}.$$

For a prime p and an integer $t\geqslant 1$, let $S_{p,t}=\{i : v_p(x_i)\geqslant t\}$, and for every nonempty $S\subseteq\{0,\dots,k\}$ put

$$y_S:=\prod_p p^{\#\{t\geqslant 1:S_{p,t}=S\}}.$$

Then

$$R=\prod_S y_S^{|S|},\quad \operatorname{lcm}(x_0,\dots,x_k)=\prod_S y_S.$$

Dropping the nesting restrictions on the sets $S_{p,t}$ only enlarges the sum, and hence

$$\sum_{x_0,\dots,x_k\geqslant 1}\frac{R^{-1/(k+1)}}{\operatorname{lcm}(x_0,\dots,x_k)}\leqslant \prod_{\emptyset\neq S\subseteq\{0,\dots,k\}}\sum_{y\geqslant 1} y^{-1-|S|/(k+1)}$$
$$=\prod_{j=1}^{k+1}\zeta\!\left(1+\frac{j}{k+1}\right)^{\binom{k+1}{j}}=:C_k.$$

This proves (2.5). Since $\zeta(1+u)\leqslant 1+u^{-1}$ for $u>0$, summing the binomial exponents gives (2.4).

For (2.6), put $n=ed$. By (2.1), $N=ng_e(n)$. Such a pair therefore satisfies $n\leqslant AX$ and $g_e(n)=N/n\geqslant 1/A$. The number of distinct values N is at most the number of such pairs (n,e), which by Markov's inequality and (2.5) is at most

$$A^k\sum_{n\leqslant AX}\sum_{\substack{e\mid n\\ e>E}} g_e(n)^k\leqslant C_kA^{k+1}E^{1-k}X.$$ □

3. Representability

Our exact identification of $\mathcal{R}$ will combine analytic ingredients and finite computations. We begin with the following lemma, which shows that almost every odd integer is represented, with a quantitative bound for the exceptional set. It is not yet sufficiently explicit for our purposes in the present section, but it already captures the spirit of the analytic reduction and it will be used for the tail estimates in the later sections.

Lemma 3.1. *There is an absolute $c>0$ such that, for $X\geqslant 3$,*

$$\#\{n\leqslant X : n \text{ odd},\ n\notin\mathcal{R}\}\ll X^{1-c}+\frac{X}{\log X}.$$

In particular, this exceptional set has size $o(X/\log\log\log X)$ as $X\to\infty$.

Proof. The theorem of Montgomery and Vaughan [13] asserts that at most $O(X^{1-c})$ even integers at most X fail to be sums of two primes. Discard also integers of the form $2p$, of which there are $O(X/\log X)$. Each remaining even integer has a representation $p+q$ with distinct primes $p<q$. The divisors of pq are $1, p, q, pq$, so $1+p+q$ is a divisor-prefix sum. □

We now prove the exact representability theorem by combining elementary constructions, finite verification, and analytic estimates. We identify the finite checks precisely and use a short rational estimate to extend the verified range.

Theorem 3.2. *The represented positive integers are precisely*

$$\mathcal{R} = \mathbb{N} \setminus \{2, 5\}.$$

In particular, $f(N)$ is defined for every $N \neq 2, 5$.

3.1. **Computational verification for finitely many integers.** Subset sums of primes in a suitable interval become divisor-prefix sums after adding 1.

Lemma 3.3. *Let $M > 1$, and suppose that $M < p_1 < \cdots < p_t < M^2$ are distinct primes. If $J \subseteq \{1, \dots, t\}$ is nonempty, then*

$$1 + \sum_{j \in J} p_j \in \mathcal{R}.$$

Proof. Put $n = \prod_{j \in J} p_j$. Every composite divisor of n is larger than M^2, whereas every prime divisor of n is less than M^2. Consequently, the positive divisors of n begin with 1, followed by the primes p_j, $j \in J$, in increasing order. Their initial sum is the displayed integer. □

Adjoining a prime extends an interval of subset sums when its translate meets or adjoins the original interval.

Lemma 3.4. *Let $C \leqslant U$ be integers. Suppose that the subset sums of a finite set of distinct primes contain every integer in $[C, U]$, and let p be a new prime satisfying $p \leqslant U - C + 1$. After adjoining p, the subset sums contain every integer in $[C, U + p]$.*

Proof. The old subset sums cover $[C, U]$, and the subset sums using p cover $[C+p, U+p]$. The hypothesis gives $C + p \leqslant U + 1$, so these two integer intervals overlap or are consecutive. □

The finite checks and interval-extension argument cover the following three overlapping ranges.

Proposition 3.5. *Every positive integer in each of the following intervals is represented:*

$$[6, 10\,000\,000], \tag{3.1}$$

$$[469\,616, 273\,803\,744\,799\,154], \tag{3.2}$$

$$[105\,000\,001, 10^{27} + 10^8]. \tag{3.3}$$

Proof. The three finite integer checks used in this proof are implemented in our verifier `verify.rs` at [9]. Its committed `output.json` records a successful run, together with interval endpoints and sample representations. The verifier constructs the representations as it runs; the JSON file does not contain all 51 000 001 representations checked in the interval $[105\,000\,000, 156\,000\,000]$.

For (3.1), take $1 \leqslant B \leqslant 1000$ and a prime $q > B$. The divisors of Bq consist first of the divisors of B and then of q times those divisors, in the same order. Thus every integer of the form

$$\sigma(B) + q \sum_{i=1}^{j} d_i(B), \quad 1 \leqslant j \leqslant \tau(B),$$

is represented. A sieve through 10 000 000 marks every integer in (3.1) except 7, which has the separate representation $7 = 1 + 2 + 4$ from $n = 4$.

For (3.2), take $M = 10\,000$ and $Y = 99\,000\,000 < M^2$. The first 94 primes in $(M, Y]$, lying between 10 007 and 10 883, have subset sums covering $[469\,615, 480\,503]$. This is checked by the exact recurrence $S_0 = \{0\}$, $S_j = S_{j-1} \cup (S_{j-1} + p_j)$ for $1 \leqslant j \leqslant 94$. The verifier checks the hypothesis of Lemma 3.4 for each successive prime through 98 999 987. The resulting subset-sum interval is

$$[469\,615, 273\,803\,744\,799\,153].$$

Lemma 3.3 then gives (3.2) after adding 1.

For (3.3), let $M = 20\,000\,000$. For every one of the 51 000 001 integers in

$$[105\,000\,000, 156\,000\,000],$$

the verifier constructs and checks a sum of four or five distinct primes in $(M, 2M)$. For completeness, the search for these representations as sums of primes is deterministic. For an odd integer, first remove the fixed prime 20 000 003; for an even integer, remove nothing. Split the resulting even number into even summands $e_1 + e_2$, beginning with the largest even e_1 not exceeding half that number and decreasing e_1 by 2, subject to $42\,000\,000 \leqslant e_i \leqslant 78\,000\,000$. For each e_i, search in increasing order for a prime p such that $p < e_i - p$, both p and $e_i - p$ lie in $(M, 2M)$, and neither has already been used. The verifier checks that this search succeeds for every integer in the interval, and then checks each resulting representation directly. The next prime after $2M$ is 40 000 003, which is at most the inclusive width 51 000 001 of the certified interval. Thus Lemma 3.4 starts the interval-extension induction. After a prime p has been adjoined, the new interval width is at least $2p$, while Bertrand's postulate supplies the next prime below $2p$; the induction therefore continues for every subsequent prime.

Set $Y = 399\,000\,000\,000\,000 < M^2$. It remains to show that

$$\sum_{2M < p \leqslant Y} p > 10^{27} + 10^8.$$

A single block suffices. Put $a = Y/2 = 199\,500\,000\,000\,000$. Dusart's explicit estimates [5, Theorem 6.9] give

$$\frac{x}{\log x} < \pi(x) \leqslant \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right) < 1.05\frac{x}{\log x} \quad (x \geqslant a),$$

because $\log a > 32$. Since $\log Y < 34$ and $2M < a$, it follows that

$$\begin{aligned}\sum_{2M < p \leqslant Y} p &\geqslant a\big(\pi(Y) - \pi(a)\big) \\ &> a\left(\frac{Y}{34} - \frac{1.05a}{32}\right) \\ &= \frac{17\,599\,173\,046\,875\,000\,000\,000\,000}{17} > 1.03 \cdot 10^{27} > 10^{27} + 10^8.\end{aligned}$$

The coarse logarithm inequalities require only $2.71 < e < 2.72$ and the rational comparisons $2.72^{32} < a$ and $Y < 2.71^{34}$. The final bound is an inequality between rational numbers, so this extension of the verified range does not depend on floating-point logarithms or on a rounding convention in the verifier. The extended subset-sum interval therefore reaches past $10^{27} + 10^8$; adding 1 and applying Lemma 3.3 proves (3.3). □

For reproducibility, we record the commands and expected output of our finite verifier. Use the following verifier [9]:
`e0e377e79538842721d9b515f53a6d821ad62cec`.
Run

```
rustc -O -C overflow-checks=on verify.rs -o /tmp/v
  /tmp/v all
```

The final JSON object should have `"check": "all"` and `"ok": true`. The integer checks use a sieve and a subset-sum recurrence to verify interval overlap and the primality, distinctness, location, and sum of each representation as a sum of primes in the third finite check. Specifically, the first check marks 9 999 994 of the 9 999 995 integers in $[6, 10^7]$, with 7 handled separately; the interval $(10\,000, 99\,000\,000]$ contains 5 705 894 primes; and the third finite check examines 51 000 001 integers. The verifier's optional numerical checks for the extension of the finite interval and for the infinite range are not needed for the rational and analytic proofs given here.

3.2. **An explicit representation by three distinct primes.**

Lemma 3.6. *Every odd integer $H \geqslant 10^{27}$ is the sum of three distinct odd primes p, q, r satisfying*

$$p, q, r > \frac{H}{30\,000 \log H}.$$

Proof. Helfgott's explicit argument [11, §7.4, equations (7.49)–(7.50)] gives a real weighted sum

$$\sum_{\substack{p+q+r=H \\ p,q,r \text{ odd primes}}} (\log p)(\log q)(\log r)\eta_+(p/x)\eta_+(q/x)\eta_*(r/x) \geqslant 0.000422H^2,$$

where $x = H/(2 + 9/(196\sqrt{2\pi}))$, for every odd $H \geqslant 10^{27}$. Here η_+ and η_* are the real smoothing functions used in that proof. Their individual smoothing factors satisfy

$$\|\eta_+\|_\infty \leqslant 1.079955, \quad \|\eta_*\|_\infty \leqslant 1.414;$$

see [11, equations (7.3) and (7.19)]. Put

$$W := (1.079955)^2 \cdot 1.414, \quad z := \frac{H}{30\,000 \log H}.$$

The explicit estimate $\psi(t) \leqslant 1.03883t$ of [16, Theorem 12] bounds the total absolute weight of triples with at least one coordinate at most z by

$$\begin{aligned} 3W\psi(z)\psi(H)\log H &\leqslant \frac{3W(1.03883)^2}{30\,000}H^2 \\ &< 0.000178H^2. \end{aligned}$$

Triples with repeated coordinates have total absolute weight at most

$$3WH(\log H)^3 < 10^{-20}H^2$$

for $H \geqslant 10^{27}$. For the last inequality, $(\log H)^3/H$ decreases on this range, and $\log(10^{27}) < 63$. Since $0.000422 > 0.000178 + 10^{-20}$, some odd-prime triple remains after both exceptional classes have been discarded. □

The preceding prime representation now proves representability throughout the remaining infinite range.

Proposition 3.7. *Every integer*

$$n \geqslant 10^{27} + 10^8$$

belongs to $\mathcal{R}$.

Proof. First let n be even and put $H = n - 1$. Apply Lemma 3.6, and order its primes as $p < q < r$. Since $H/(\log H)^2$ increases for $H > e^2$ and

$$\frac{10^{27}}{(30\,000 \log(10^{27}))^2} > 1,$$

we have

$$pq > \left(\frac{H}{30\,000\log H}\right)^2 > H > r.$$

Hence the divisors of pqr begin with $1, p, q, r$, and their sum is $1 + p + q + r = n$.

Now let n be odd. By Bertrand's postulate, choose an odd prime ℓ such that

$$60\,000\log n < \ell < 120\,000\log n,$$

and put $H = n - 1 - \ell$. Set $T = 10^{27} + 10^8$. The function

$$h(t) := t - 1 - 120\,000\log t - 10^{27}$$

has positive derivative for $t \geqslant T$, and $h(T) > 0$ because $\log T < 64$. Therefore $H \geqslant 10^{27}$; in the same way, $H > n/2$. Apply Lemma 3.6 and order the resulting primes as $p < q < r$. Since $\log H < \log n$,

$$p > \frac{H}{30\,000\log H} > \frac{n}{60\,000\log n} > 120\,000\log n > \ell.$$

For the penultimate inequality, observe that $n/(\log n)^2$ increases for $n > e^2$ and already exceeds $7.2 \cdot 10^9$ at $n = T$. Finally,

$$\ell p > 60\,000\log n \cdot \frac{H}{30\,000\log H} > 2H > r.$$

Thus the divisors of ℓpqr begin with $1, \ell, p, q, r$, whose sum is $1 + \ell + p + q + r = n$. □

Proof of Theorem 3.2. The three intervals of Proposition 3.5 overlap, and the last one overlaps the infinite interval of Proposition 3.7 at $10^{27} + 10^8$. Hence every $n \geqslant 6$ is represented. The integers $1, 3, 4$ are represented by $m = 1, 2, 3$, respectively. No prefix can equal 2: its first term is 1, and every longer prefix has sum at least $1 + 2 = 3$. Similarly, a two-term prefix equal to 5 would have to be $1 + 4$, but $4 \mid m$ implies $2 \mid m$, contradicting that 4 is the second-smallest divisor. Every prefix of length at least three has sum at least $1 + 2 + 3 = 6$. Therefore neither 2 nor 5 is represented. □

4. Lower tails

4.1. **Upper bounds for the lower tail.** We extend the first-moment argument from Tao's comment on [2] to obtain the following higher-moment estimate.

Lemma 4.1. *There is an absolute constant $C > 0$ such that, for every integer $q \geqslant 3$ and every real $x \geqslant 1$,*

$$\sum_{n \leqslant x} \sum_{j \geqslant 0} \left(\frac{\sigma_j(n)}{n}\right)^q \ll \exp(Cq\log\log q)x.$$

The implicit constant is absolute.

Proof. Let

$$1 = r_1 < r_2 < \cdots < r_{\tau(n)} = n$$

be the divisors of n. The substitution $r \mapsto n/r$ gives

$$\frac{\sigma_j(n)}{n} = \sum_{i > j} \frac{1}{r_i}.$$

After expanding the qth power and summing over j, a fixed tuple $a_1, \ldots, a_q$ of divisors is counted once for each divisor threshold not exceeding $\min_i a_i$. Hence

$$\sum_{j \geqslant 0} \left(\frac{\sigma_j(n)}{n}\right)^q = \sum_{\substack{r, a_1, \ldots, a_q \mid n \\ r \leqslant \min(a_1, \ldots, a_q)}} \frac{1}{a_1 \cdots a_q}.$$

Summing over $n \leqslant x$ and bounding the number of multiples of $\operatorname{lcm}(r, a_1, \dots, a_q)$ by $x/\operatorname{lcm}(r, a_1, \dots, a_q)$, it is enough to prove

$$S := \sum_{\substack{r, a_1, \dots, a_q \geqslant 1 \\ r \leqslant \min_i a_i}} \frac{1}{a_1 \cdots a_q \operatorname{lcm}(r, a_1, \dots, a_q)} \ll \exp(Cq \log\log q). \tag{4.1}$$

For each i, put

$$b_i := \frac{r}{\gcd(r, a_i)}, \quad c_i := \frac{a_i}{\gcd(r, a_i)}.$$

Then $b_i \mid r$, $\gcd(b_i, c_i) = 1$, $a_i = rc_i/b_i$, and

$$\operatorname{lcm}(r, a_1, \dots, a_q) = r \operatorname{lcm}(c_1, \dots, c_q).$$

The condition $r \leqslant a_i$ becomes $b_i \leqslant c_i$. Dropping the coprimality conditions and taking $\beta = (q-1)/q$, the inequality

$$\frac{1}{c_i} \leqslant \frac{1}{b_i^{1-\beta} c_i^{\beta}}$$

for $c_i \geqslant b_i$ applies term by term. Replacing each divisor $b_i \mid r$ by r/d_i and using $\beta q = q - 1$ leaves the power r^{-2}. Thus $S \leqslant S'S''$, where

$$S' := \sum_{r \geqslant 1} \frac{1}{r^2} \left(\sum_{d \mid r} \frac{1}{d^{(q-1)/q}} \right)^q$$

and

$$S'' := \sum_{c_1, \dots, c_q \geqslant 1} \frac{1}{c_1^{(q-1)/q} \cdots c_q^{(q-1)/q} \operatorname{lcm}(c_1, \dots, c_q)}.$$

Both sums have Euler product representations.

For S', write $S' = \prod_p L_p$, where

$$L_p := \sum_{\ell \geqslant 0} p^{-2\ell} \left(\sum_{0 \leqslant \gamma \leqslant \ell} p^{-((q-1)/q)\gamma} \right)^q.$$

For $p \leqslant q$,

$$L_p \leqslant (1 - p^{-2})^{-1} (1 - p^{-(q-1)/q})^{-q}.$$

Since

$$\sum_{p \leqslant q} p^{-(q-1)/q} \leqslant q^{1/q} \sum_{p \leqslant q} \frac{1}{p} \ll \log\log q,$$

one obtains

$$\prod_{p \leqslant q} L_p \ll \exp(Cq \log\log q).$$

For $p > q$, one has $qp^{-(q-1)/q} \ll 1$, and the terms with $\ell \geqslant 1$ give $L_p = 1 + O(p^{-2})$. Thus

$$S' \ll \exp(Cq \log\log q). \tag{4.2}$$

For S'', write $S'' = \prod_p M_p$, where

$$M_p := \sum_{\gamma_1, \dots, \gamma_q \geqslant 0} p^{-((q-1)/q)(\gamma_1 + \cdots + \gamma_q) - \max_i \gamma_i}.$$

Put

$$A_h := \sum_{0 \leqslant \gamma \leqslant h} p^{-((q-1)/q)\gamma}.$$

Grouping by $h = \max_i \gamma_i$ gives

$$M_p = \sum_{h \geqslant 0} p^{-h}(A_h^q - A_{h-1}^q),$$

where $A_{-1} := 0$. For $p \leqslant q$,

$$M_p \leqslant (1-p^{-1})^{-1}(1-p^{-(q-1)/q})^{-q},$$

so the product over these primes is $\ll \exp(Cq \log\log q)$. For $p > q$, the mean-value theorem and $qp^{-(q-1)/q} \ll 1$ give

$$A_h^q - A_{h-1}^q \ll qp^{-h(q-1)/q}.$$

Consequently,

$$M_p - 1 \ll qp^{-2+1/q}, \quad \sum_{p>q}(M_p - 1) \ll q\sum_{n>q} n^{-2+1/q} \ll 1.$$

Hence

$$S'' \ll \exp(Cq \log\log q). \tag{4.3}$$

Combining (4.2) and (4.3) proves (4.1) and the lemma. □

Proof of Theorem 1.1. Let

$$\mathcal{E}_\delta(X) := \{N \leqslant X : N \in \mathcal{R},\ f(N) \leqslant \delta N\}.$$

If $\delta X < 1$, then $\mathcal{E}_\delta(X)$ is empty, so assume $\delta X \geqslant 1$. If $N \in \mathcal{E}_\delta(X)$ and $n = f(N)$, then $n \leqslant \delta X$ and $N = \sigma_j(n)$ for some $j \geqslant 0$ with $\sigma_j(n)/n \geqslant 1/\delta$. Therefore, for every integer $q \geqslant 3$,

$$\begin{aligned}\#\mathcal{E}_\delta(X) &\leqslant \delta^q \sum_{n \leqslant \delta X}\sum_{j \geqslant 0}\left(\frac{\sigma_j(n)}{n}\right)^q \\ &\ll \delta^{q+1}\exp(Cq\log\log q)X.\end{aligned}$$

Choose a sufficiently small absolute $c > 0$ and put

$$q := \lfloor \exp((1/\delta)^c) \rfloor .$$

For all sufficiently small δ, one has $C\log\log q \leqslant \frac{1}{2}\log(1/\delta)$, and hence

$$\delta^{q+1}\exp(Cq\log\log q) \leqslant \exp\left(-\frac{q}{2}\log\frac{1}{\delta}\right) \leqslant \exp\left(-\exp((1/\delta)^{c/2})\right).$$

Renaming $c/2$ as c proves the first assertion. The fixed-power bound follows because this double-exponential expression is $O_M(\delta^M)$ for every fixed $M > 0$. □

4.2. **Apparent absence of small ratios.** The positive-density assertion for every fixed small ratio is compatible with the absence of such ratios at any practical search range. The following bound makes this claim quantitative.

Proposition 4.2. *If $n \in \mathcal{R}$ and $f(n) \leqslant n/10$, then $n > 10^{120}$.*

Proof. Put $m = f(n)$. Every divisor-prefix sum of m is at most $\sigma(m)$, so $f(n) \leqslant n/10$ implies

$$\frac{\sigma(m)}{m} \geqslant \frac{n}{m} \geqslant 10.$$

If $m \leqslant 5040$, then

$$\frac{\sigma(m)}{m} = \sum_{d|m}\frac{1}{d} \leqslant \sum_{d=1}^{5040}\frac{1}{d} \leqslant 1 + \log 5040 < 10,$$

a contradiction. If $n \leqslant 10^{120}$, then $m \leqslant 10^{119}$. Hence Axler's unconditional refinement of Robin's inequality [1, Corollary 2] gives

$$\frac{\sigma(m)}{m} < (1 + 3.15367\cdot 10^{-7})e^\gamma \log\log m \leqslant 9.99745 < 10,$$

again a contradiction. □

The obstruction is not vacuous. For $m = \operatorname{lcm}(1, \dots, 289)$, direct evaluation of the finite Euler product for $\sigma(m)/m$ gives

$$\frac{\sigma(m)}{m} = 10.004735741\ldots > 10.$$

Thus $n = \sigma(m)$ satisfies

$$n = 3.078152209\ldots \cdot 10^{128}, \quad \frac{f(n)}{n} \leqslant \frac{m}{\sigma(m)} = 0.099952665\ldots < \frac{1}{10}.$$

4.3. **A family of integers with large divisor sums.** We now prove Theorem 1.2 by adapting the positive-density range argument of Luca and Pomerance [12]. The main additional requirement is that $\sigma(n)/n$ exceed a prescribed constant throughout the chosen family. We ensure this by requiring a fixed divisor D. A theorem of Pollack [14] then bounds $\sigma(s(n))/s(n)$ for the intermediate integers n; this replaces the deficiency condition used in [12]. After imposing the required divisibility conditions, we partition the family according to the largest Y-smooth divisor and bound the number of pairs with the same aliquot sum. The Cauchy–Schwarz inequality will then give a positive lower bound for the density of the represented values.

If $n \geqslant 2$, then $s(n)$ is the sum of all proper divisors of n, so $s(n) \in \mathcal{R}$ and

$$f(s(n)) \leqslant n. \tag{4.4}$$

We will construct $\mathcal{A}(X) \subset [1, X]$ such that

$$n/\delta < s(n) \leqslant C_\delta X$$

for $n \in \mathcal{A}(X)$ and

$$\#s(\mathcal{A}(X)) \gg_\delta X.$$

By (4.4), rescaling X then proves the theorem.

It is enough to prove the theorem for $0 < \delta \leqslant 1$. Put $\lambda := 1/\delta$, and choose a fixed squarefree integer D such that

$$\frac{\sigma(D)}{D} > 1 + \lambda. \tag{4.5}$$

Such a D exists because $\prod_p (1 + 1/p) = \infty$.

For large X, let $\mathcal{A}_0(X)$ consist of the integers

$$M = pqrk \tag{4.6}$$

for which p, q, r are prime,

$$X^{1/120} < k = Dj \leqslant X^{1/60}, \quad \frac{\sigma(j)}{j} \leqslant 4\zeta(2),$$
$$X^{1/15} < r \leqslant X^{1/12}, \quad X^{7/20} < q \leqslant X^{11/30}, \quad \frac{X}{2qrk} < p \leqslant \frac{X}{qrk}.$$

Set $\ell := rk$ and $n := q\ell$.

Lemma 4.3. *There is a constant $C_\delta > 0$ such that, for all sufficiently large X,*

$$\#\mathcal{A}_0(X) \gg_\delta X,$$

every $M \in \mathcal{A}_0(X)$ has a unique factorization (4.6)*, and*

$$\lambda < \frac{s(t)}{t} < C_\delta \quad (t \in \{k, \ell, n, M\}). \tag{4.7}$$

Moreover, $q > n^{7/9}$.

Proof. For $J \geqslant 1$,

$$\sum_{j \leqslant J} \frac{\sigma(j)}{j^2} = \sum_{d \leqslant J} \frac{1}{d^2} \sum_{a \leqslant J/d} \frac{1}{a} \leqslant \zeta(2) \sum_{a \leqslant J} \frac{1}{a}.$$

Consequently, with $J_1 = X^{1/120}/D$ and $J_2 = X^{1/60}/D$,

$$\sum_{\substack{J_1 < j \leqslant J_2 \\ \sigma(j)/j \leqslant 4\zeta(2)}} \frac{1}{j} \geqslant \sum_{J_1 < j \leqslant J_2} \frac{1}{j} - \frac{1}{4\zeta(2)} \sum_{j \leqslant J_2} \frac{\sigma(j)}{j^2} \geqslant \left(\frac{1}{240} + o(1) \right) \log X.$$

For fixed q, r, k, the interval for p is $(T/2, T]$, where $T = X/(qrk) \geqslant X^{8/15}$. The prime number theorem, followed by Mertens' theorem, therefore gives

$$\#\mathcal{A}_0(X) \gg \frac{X}{\log X} \sum_k \frac{1}{k} \sum_{X^{1/15} < r \leqslant X^{1/12}} \frac{1}{r} \sum_{X^{7/20} < q \leqslant X^{11/30}} \frac{1}{q} \gg_\delta X.$$

The factorization is unique because $p > X^{1/2}$, the prime q lies above $X^{7/20}$, the prime r lies between $X^{1/15}$ and $X^{1/12}$, and every prime factor of k is at most $X^{1/60}$.

The function $n \mapsto \sigma(n)/n$ is nondecreasing under divisibility. Since $D \mid t$ for $t \in \{k, \ell, n, M\}$, (4.5) therefore gives the lower bound in (4.7). The inequality $\sigma(ab)/(ab) \leqslant (\sigma(a)/a)(\sigma(b)/b)$ also gives

$$\frac{\sigma(k)}{k} \leqslant \frac{\sigma(D)}{D} \frac{\sigma(j)}{j} \leqslant 4\zeta(2) \frac{\sigma(D)}{D},$$

and adjoining each of p, q, r multiplies $\sigma(t)/t$ by less than 2. This gives the upper bound in (4.7). Finally, $\ell \leqslant X^{1/10}$ and $q^2 > X^{7/10} \geqslant \ell^7$, whence $q^9 > q^7 \ell^7 = n^7$. □

4.4. **Divisibility properties and the largest smooth divisor.** For u sufficiently large, put

$$y(u) := \frac{\log\log u}{\log\log\log u}, \quad Y := y(X^{1/120}).$$

We collect the divisibility properties of divisor sums that Luca and Pomerance established for almost all integers.

Lemma 4.4. *Outside a set of integers of asymptotic density zero, the following hold:*

(i) $v_p(\sigma(n)) > v_p(n)$ *for every prime* $p \leqslant y(n)$*;*
(ii) $P^+(\gcd(n, \sigma(n))) \leqslant y(n)$*;*
(iii) $\sigma(n)/\gcd(n, \sigma(n))$ *is divisible by every prime* $p \leqslant y(n)$*;*
(iv) *every prime factor of* $s(n)/\gcd(n, \sigma(n))$ *exceeds* $y(n)$*.*

If in addition $P^+(n) > n^{7/9}$*, then, outside another density-zero set,*

$$y(n) < q_0 \leqslant n^{10/27}, \quad q_0 \text{ prime} \quad \Longrightarrow \quad q_0^2 \nmid s(n).$$

Finally, outside a set of asymptotic density zero,

$$\sum_{\substack{p \mid \sigma(n) \\ p > (\log\log n)^2 \\ p \text{ prime}}} \frac{1}{p} \leqslant 1.$$

Proof. The first four properties are the conclusions of [12, Lemma 2.1], with the valuation inequality in (i) stated explicitly in the proof of its part (4). The square-divisibility assertion is the consequence of [12, Lemma 2.2] needed here; we use it only in the displayed range. The estimate for reciprocals of prime divisors is [12, Lemma 2.5]. □

We next count integers with a prescribed smooth part and bound the proportion whose smooth part is large.

Lemma 4.5. *Suppose that $Y = Y(X) \to \infty$ and $Y = o(\log X)$. For every fixed $c > 0$, uniformly for Y-smooth $d \leqslant Y^c$,*

$$\#\{n \leqslant X : d \text{ is the largest } Y\text{-smooth divisor of } n\} \asymp \frac{X}{d \log Y}.$$

Moreover, given $\eta > 0$, c may be chosen sufficiently large that the integers $n \leqslant X$ whose largest Y-smooth divisor exceeds Y^c account for at most ηX integers for all sufficiently large X.

Proof. Write $D_Y(n)$ for the largest Y-smooth divisor of n. For a Y-smooth integer d, $D_Y(n) = d$ if and only if $n = du$ with $\gcd(u, Y\#) = 1$. Counting complete periods modulo $Y\#$ gives

$$\#\{n \leqslant X : D_Y(n) = d\} = \frac{X}{d}\Delta(Y) + O(Y\#).$$

Chebyshev's estimate gives $Y\# = \exp(O(Y)) = X^{o(1)}$, while Mertens' theorem gives $\Delta(Y) \asymp 1/\log Y$. This proves the required uniform estimate. It is the small-Y form of the counting argument in [12, (3.1)].

To bound the number of integers with a large Y-smooth divisor, sum $\log D_Y(n)$:

$$\sum_{n \leqslant X} \log D_Y(n) = \sum_{p \leqslant Y} \log p \sum_{a \geqslant 1} \left\lfloor \frac{X}{p^a} \right\rfloor \leqslant X \sum_{p \leqslant Y} \frac{\log p}{p-1} \ll X \log Y.$$

The last estimate follows by partial summation from Chebyshev's bound. Markov's inequality now gives

$$\#\{n \leqslant X : D_Y(n) > Y^c\} \ll X/c.$$

Choosing c sufficiently large in terms of η proves the assertion. □

The following lemma selects a large subfamily satisfying the divisibility conditions and estimates needed for the second moment.

Lemma 4.6. *There is a subfamily $\mathcal{A}(X) \subseteq \mathcal{A}_0(X)$ with $\#\mathcal{A}(X) \gg_\delta X$ having the following properties. For $M = pqrk \in \mathcal{A}(X)$, let d be the largest Y-smooth divisor of M. Then, for every $t \in \{k, \ell, n, M\}$,*

(i) *$d = \gcd(t, \sigma(t))$, and d is the largest Y-smooth divisor of both t and $s(t)$;*
(ii) *$\sigma(t)/d$ is divisible by every prime at most Y, while every prime factor of $s(t)/d$ exceeds $y(t)$.*

In addition,

$$y(n) < q_0 \leqslant n^{10/27}, \quad q_0 \text{ prime} \quad \Longrightarrow \quad q_0^2 \nmid s(n),$$

and

$$\sum_{\substack{p \mid \sigma(n) \\ p > (\log\log X)^2 \\ p \text{ prime}}} \frac{1}{p} \leqslant 1. \tag{4.8}$$

There is also a constant $B_\delta > 0$ such that

$$\frac{\sigma(s(n))}{s(n)} \leqslant B_\delta. \tag{4.9}$$

Proof. First discard the integers for which k has a prime factor in $(Y, y(X)]$. Uniformly for $X^{1/120} < k \leqslant X^{1/60}$,

$$y(k) = y(X) + O\left(\frac{1}{\log\log\log X}\right),$$

so

$$\sum_{Y < p \leqslant y(X)} \frac{1}{p} = o(1).$$

Indeed,

$$\sum_{\substack{k\leqslant X^{1/60}\\ p\mid k \text{ for some } Y<p\leqslant y(X)}} \frac{1}{k} \ll \log X \sum_{Y<p\leqslant y(X)} \frac{1}{p} = o(\log X).$$

The discarded k therefore account for $o(X)$ members of $\mathcal{A}_0(X)$.

Apply Lemma 4.4 to each of k, ℓ, n, M, with its large-prime square divisibility conclusion applied to n.

Each excluded set has asymptotic density zero. These exclusions remove only $o(X)$ members of $\mathcal{A}_0(X)$. To justify this assertion, let $\mathcal{E} \subseteq \mathbb{N}$ satisfy $\#(\mathcal{E}\cap[1,T]) = o(T)$. Partial summation gives

$$\sum_{\substack{n\leqslant T\\ n\in\mathcal{E}}} \frac{1}{n} = o(\log T). \tag{4.10}$$

The separated factor ranges make the decompositions $\ell = rk$, $n = q\ell$ and $M = pn$ unique. An upper-bound estimate for the prime p then shows that the members for which one of k, ℓ, n lies in $\mathcal{E}$ are

$$\ll \frac{X}{\log X} \sum_{\substack{t\leqslant X^{O(1)}\\ t\in\mathcal{E}}} \frac{1}{t} = o(X),$$

up to bounded reciprocal prime sums; the deletion at the level M is directly $o(X)$.

For every retained t, Lemma 4.4 and its first four properties identify $\gcd(t, \sigma(t))$ with the largest $y(t)$-smooth divisor of t and give the corresponding divisibility properties of $\sigma(t)$ and $s(t)$. Since $Y \leqslant y(t) \leqslant y(X)$, the exclusion made in the first paragraph and the fact that $p, q, r > y(X)$ show that the same integer d occurs at all four levels and prove (i)–(ii). The hypothesis $q > n^{7/9}$ gives the displayed square-divisibility exclusion for $s(n)$, and the final assertion of Lemma 4.4 gives (4.8), since $n \leqslant X$ and increasing the threshold only decreases the sum.

By [14, Theorem 1.4], the fixed exceptional set of integers n for which $s(s(n))/s(n) > s(n)/n + 1$ has asymptotic density zero. One further exclusion at the level n, justified by (4.10), therefore ensures that

$$\frac{s(s(n))}{s(n)} \leqslant \frac{s(n)}{n} + 1.$$

Together with (4.7), this yields (4.9) with $B_\delta := C_\delta + 2$. □

Partition $\mathcal{A}(X)$ according to the integer d in Lemma 4.6. The image sets belonging to distinct classes are disjoint, because d is also the largest Y-smooth divisor of $s(M)$.

Lemma 4.7. *There are positive constants c, c_1, c_2, depending at most on δ, and a set $\mathcal{D}_X$ of Y-smooth integers $d \leqslant Y^c$ such that, writing*

$$\mathcal{A}_d(X) := \{M \in \mathcal{A}(X) : d \text{ is the largest } Y\text{-smooth divisor of } M\},$$

one has, uniformly for $d \in \mathcal{D}_X$,

$$c_1 \frac{X}{d\log Y} \leqslant \#\mathcal{A}_d(X) \leqslant c_2 \frac{X}{d\log Y}, \quad \sum_{d\in\mathcal{D}_X} \frac{1}{d} \gg_\delta \log Y. \tag{4.11}$$

The sets $s(\mathcal{A}_d(X))$, $d \in \mathcal{D}_X$, are pairwise disjoint.

Proof. Fix $\eta = \eta(\delta) > 0$ with $\#\mathcal{A}(X) \geqslant \eta X$. Apply Lemma 4.5, choosing c so large that the classes with $d > Y^c$ contain at most $\eta X/3$ integers. Since

$$\sum_{P^+(d)\leqslant Y} \frac{1}{d} = \prod_{p\leqslant Y} \left(1 - \frac{1}{p}\right)^{-1} \asymp \log Y,$$

we may choose a sufficiently small $a = a(\delta) > 0$ and let $\mathcal{D}_X$ consist of those $d \leqslant Y^c$ for which

$$\#\mathcal{A}_d(X) \geqslant a\frac{X}{d\log Y}.$$

The classes with $d \leqslant Y^c$ outside $\mathcal{D}_X$ then contain at most $\eta X/3$ integers. Thus the classes in $\mathcal{D}_X$ contain $\gg_\delta X$ integers. The uniform upper estimate above now gives

$$\frac{X}{\log Y}\sum_{d\in\mathcal{D}_X}\frac{1}{d} \gg_\delta X,$$

which proves (4.11). The disjointness assertion follows from Lemma 4.6. □

4.5. **The second-moment estimate.**

Proposition 4.8. *For $d \in \mathcal{D}_X$, put*

$$R_d(u) := \#\{M \in \mathcal{A}_d(X) : s(M) = u\}.$$

Then, uniformly for $d \in \mathcal{D}_X$,

$$\sum_u R_d(u)^2 \ll_\delta \frac{X}{d\log Y}. \tag{4.12}$$

Proof. We prove the second-moment estimate directly, following the organization of [12, (3.4), §§3.1–3.2]. All estimates needed for the present family are recorded below. We first note two reciprocal estimates. Since d is the largest Y-smooth divisor of every member of $\mathcal{A}_d(X)$, the corresponding factor k is d times an integer with no prime factor at most Y. The complete-period count in Lemma 4.5, applied uniformly throughout the fixed power interval for k, and partial summation give

$$\sum_k \frac{1}{k} \ll \frac{\log X}{d\log Y}. \tag{4.13}$$

Here and below a sum over a factor of a member of $\mathcal{A}_d(X)$ is restricted to the ranges in (4.6). The reciprocal prime sums over the fixed power intervals for q and r are bounded, so

$$\sum_n \frac{1}{n} \ll \frac{\log X}{d\log Y}. \tag{4.14}$$

The diagonal contribution to $\sum_u R_d(u)^2$ is $\#\mathcal{A}_d(X)$ and is bounded by (4.11). Consider a pair $M = pn$, $M' = p'n'$ with $s(M) = s(M')$ and $n \neq n'$. Then

$$p\,s(n) + \sigma(n) = p'\,s(n') + \sigma(n'). \tag{4.15}$$

By Lemma 4.6, $\gcd(n, \sigma(n)) = \gcd(n', \sigma(n')) = d$. Put

$$h := \frac{\gcd(s(n), s(n'))}{d}.$$

Every prime factor of h exceeds both $y(n)$ and $y(n')$. Equation (4.15) shows that dh divides $\sigma(n) - \sigma(n')$. This difference is nonzero. Indeed, equality would give $ps(n) = p's(n')$; for all sufficiently large X,

$$\min(p, p') > \tfrac{1}{2}X^{8/15} > C_\delta X^{7/15} \geqslant \max\big(s(n), s(n')\big),$$

which forces $p = p'$, then $s(n) = s(n')$, and finally $n = n'$. Thus

$$A_1 := \frac{s(n)}{dh}, \quad A_2 := \frac{s(n')}{dh}, \quad A_3 := \frac{|\sigma(n) - \sigma(n')|}{dh}$$

are positive integers, with $\gcd(A_1, A_2) = 1$.

For fixed n, n', elementary linear Diophantine theory shows that there are fixed integers p_0, p_0' for which every integral solution has the form

$$p = p_0 + A_2 t, \quad p' = p_0' + A_1 t$$

for $t \in \mathbb{Z}$. All relevant values of t lie in an interval of length at most

$$T := \frac{Xdh}{ns(n')}.$$

The bounds

$$\lambda n' \leqslant s(n') \leqslant C_\delta n', \quad nn' \leqslant X^{14/15},$$

give $T \gg_\delta X^{1/15}$ and $T \ll_\delta Xdh/(nn')$, so $\log T \asymp \log X$. The affine forms are primitive whenever a prime pair exists. For example, a common divisor of p_0 and A_2 would divide every value of $p = p_0 + A_2 t$, whereas $p > A_2$ for all sufficiently large X; the second form is identical. For odd primes the two primitive forms exclude at most two residue classes, and the prime 2 either leaves an allowed parity class or there are no solutions. The standard two-dimensional upper-bound sieve, in the form used in [12, (3.7)], therefore gives

$$\ll_\delta \frac{Xdh}{nn'(\log X)^2} \frac{A_1}{\varphi(A_1)} \frac{A_2}{\varphi(A_2)} \frac{A_3}{\varphi(A_3)}$$

possible prime pairs.

If $v \mid s(n)$, then

$$\frac{v}{\varphi(v)} \leqslant \zeta(2)\frac{\sigma(v)}{v} \leqslant \zeta(2)\frac{\sigma(s(n))}{s(n)} \leqslant \zeta(2)B_\delta \tag{4.16}$$

and the same holds for n' in place of n. This bounds the factors belonging to A_1, A_2, and also gives $\varphi(h) \gg_\delta h$.

We now reduce the factor belonging to A_3. Let $A_{3,1}$ contain its prime factors at most $(\log\log X)^2$, let $A_{3,2}$ contain those in $((\log\log X)^2, \log X]$, and let $A_{3,3}$ contain those exceeding $\log X$. Since $A_3 \ll_\delta X^{7/15}$,

$$\frac{A_{3,1}}{\varphi(A_{3,1})} \ll \log Y, \quad \frac{A_{3,3}}{\varphi(A_{3,3})} \ll 1.$$

Let $A'_{3,2}$ be the largest divisor of $A_{3,2}$ coprime to $\sigma(n)$. The primes removed from $A_{3,2}$ divide $\sigma(n)$ and exceed $(\log\log X)^2$. Since $-\log(1-1/p) \leqslant 2/p$ for every prime p, (4.8) gives

$$\frac{A_{3,2}/\varphi(A_{3,2})}{A'_{3,2}/\varphi(A'_{3,2})} = \prod_{\substack{p \mid A_{3,2} \\ p \mid \sigma(n) \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right)^{-1} \leqslant \exp\left(2 \sum_{\substack{p \mid \sigma(n) \\ p > (\log\log X)^2 \\ p \text{ prime}}} \frac{1}{p}\right) \leqslant e^2.$$

Consequently, it remains to prove

$$\frac{X \log Y}{(\log X)^2} \sum_{n \neq n'} \frac{dh}{nn'} \frac{A'_{3,2}}{\varphi(A'_{3,2})} \ll_\delta \frac{X}{d \log Y}, \tag{4.17}$$

where the sum is restricted to pairs $n \neq n'$ for which (4.15) has a solution in primes p, p' in the prescribed ranges, and $dh = \gcd(s(n), s(n'))$. In particular, $dh \mid \sigma(n) - \sigma(n')$. Notice also that A_3 is divisible by every prime at most Y. Indeed, both $\sigma(n)/d$ and $\sigma(n')/d$ have this property by Lemma 4.6, while every prime factor of h exceeds Y.

Suppose first that $h > X^{10/33}$. Write $n = q\ell$ and $n' = q'\ell'$. We claim that $\gcd(h, s(\ell)\sigma(\ell)) = 1$. If a prime q_0 divided both h and $\sigma(\ell)$, then the identity $s(n) = qs(\ell) + \sigma(\ell)$ would imply either $q_0 = q$ or $q_0 \mid s(\ell)$. In the first case, $q_0 \mid \gcd(n, \sigma(n)) = d$; in the second case, $q_0 \mid \ell$ and $q_0 \mid \sigma(\ell)$, so again $q_0 \mid d$. Both conclusions contradict the fact that every prime factor of h exceeds Y, whereas d is Y-smooth. If instead $q_0 \mid h$ and $q_0 \mid s(\ell)$, then the same identity gives $q_0 \mid \sigma(\ell)$, which has just been excluded. Thus both $s(\ell)$ and $\sigma(\ell)$ are units modulo h. It follows that q is fixed in a reduced residue class modulo h, and the same holds for q'. Moreover,

$$n \equiv \sigma(n) \equiv \sigma(n') \equiv n' \pmod{h}.$$

Eliminating q, q' gives

$$s(\ell')\ell\sigma(\ell) - s(\ell)\ell'\sigma(\ell') \equiv 0 \pmod{h}. \tag{4.18}$$

The absolute value of the left side is at most

$$2C_\delta(C_\delta+1)\max(\ell,\ell')^3 \ll_\delta X^{3/10} = o(X^{10/33}),$$

so it vanishes. For every prime power $q_0^a \| d$, Lemma 4.6 gives $q_0^a \| s(\ell)$ and $q_0^{a+1} \mid \sigma(\ell)$; it also excludes primes exceeding Y from $\gcd(\ell, s(\ell))$. Hence

$$\gcd(\ell^2, s(\ell)) = d, \quad \gcd((\ell')^2, s(\ell')) = d.$$

The equality in (4.18) can be written as

$$\frac{\ell^2}{s(\ell)} + \ell = \frac{(\ell')^2}{s(\ell')} + \ell'.$$

The reduced denominators of the two fractions are $s(\ell)/d$ and $s(\ell')/d$. Since their difference is an integer, these denominators are equal. Thus $s(\ell) = s(\ell')$, and the displayed equality, viewed as a strictly increasing quadratic in ℓ, gives $\ell = \ell'$.

In this range Mertens' theorem gives $A'_{3,2}/\varphi(A'_{3,2}) \ll \log\log X/\log Y$. Summing the left side of (4.17) and using $\ell = \ell'$ gives

$$\ll \frac{Xd\log\log X}{(\log X)^2} \sum_{q,q',\ell,h} \frac{h}{qq'\ell^2}.$$

We may assume $q > q'$ by symmetry. Because $\ell = \ell'$, the two primes belong to the same residue class modulo h. Thus $q = q' + jh$ with $j \geqslant 1$, and

$$\sum_q \frac{1}{q} \leqslant \sum_{1\leqslant j\leqslant X} \frac{1}{jh} \ll \frac{\log X}{h},$$

even if primality is discarded. Summing next over $h \mid s(q'\ell)$ is bounded by $\tau(s(q'\ell)) = X^{o(1)}$. The reciprocal sum over q' is bounded and $\sum_\ell \ell^{-2} \ll X^{-1/15}$. The contribution is therefore

$$dX^{14/15+o(1)} = o\left(\frac{X}{d\log Y}\right),$$

uniformly for $d \leqslant Y^c$.

It remains to treat $h \leqslant X^{10/33}$. Let $A''_{3,2}$ be the largest divisor of $A'_{3,2}$ coprime to $s(n')$. Equation (4.16) gives

$$\frac{A'_{3,2}}{\varphi(A'_{3,2})} \ll_\delta \frac{A''_{3,2}}{\varphi(A''_{3,2})}.$$

We first note that h is squarefree in this range. Indeed, if $q_0^2 \mid h$, then $q_0^2 \mid s(n)$ and $q_0 > y(n)$. By Lemma 4.4, this forces $q_0 > n^{10/27}$. Since $n > X^{17/40}$,

$$q_0^2 > n^{20/27} > X^{17/54} > X^{10/33} \geqslant h,$$

a contradiction.

Fix n', h, k. The congruences

$$qrk \equiv (q+1)(r+1)\sigma(k) \equiv n' \pmod{h}$$

may therefore be solved for the required symmetric functions. Indeed, any prime dividing both h and either n or $\sigma(n)$ would divide $\gcd(n, \sigma(n)) = d$, which is impossible because every prime factor of h exceeds Y. Thus k and $\sigma(k)$ are units modulo h. The two congruences therefore determine qr and $q + r$ modulo h. Since h is squarefree, there are at most $\tau(h)$ possible ordered residue pairs

$$q \equiv a \pmod{h}, \quad r \equiv b \pmod{h}.$$

For one such pair put

$$\mathfrak{f}(qr) := \sum_{\substack{(\log\log X)^2 < q_0 \leqslant \log X \\ q_0 \mid \sigma(kqr) - \sigma(n') \\ q_0 \nmid h\sigma(kqr)}} \frac{1}{q_0},$$

where q_0 is prime. Mertens' theorem gives

$$\frac{A''_{3,2}}{\varphi(A''_{3,2})} \ll_\delta 1 + \frac{\log\log X}{\log Y}\mathfrak{f}(qr). \tag{4.19}$$

Indeed, when $\mathfrak{f}(qr) \leqslant 1$, the logarithm of the left side is $O(\mathfrak{f}(qr))$. When $\mathfrak{f}(qr) > 1$, Mertens' theorem bounds the contribution of the full prime range by $O(\log\log X/\log Y)$.

For fixed n', h, k, r and a prime q_0 occurring in $\mathfrak{f}(qr)$, the congruence

$$(q+1)\sigma(kr) \equiv \sigma(n') \pmod{q_0}$$

fixes q modulo q_0. The condition $q_0 \nmid \sigma(kqr)$ ensures that $\sigma(kr)$ is invertible modulo q_0. Since $q_0 \nmid h$, this congruence combines with $q \equiv a \pmod h$. Brun–Titchmarsh and partial summation, followed by (4.16), give

$$\sum_{q,r} \frac{\mathfrak{f}(qr)}{qr} \ll_\delta \begin{cases} \dfrac{\log X}{hX^{1/20}}, & X^{1/20} < h \leqslant X^{10/33}, \\ \dfrac{1}{h^2(\log\log X)^2}, & h \leqslant X^{1/20}. \end{cases} \tag{4.20}$$

Here the sums are over the fixed residue pair. To see (4.20), first note that a nonreduced residue class modulo q_0 contains no admissible q, because $q > X^{7/20} > \log X \geqslant q_0$. Otherwise the combined class modulo q_0h is reduced. Uniformly in the present ranges, $q_0h \leqslant X^{10/33}\log X < X^{7/20-\epsilon_0}$ for a fixed $\epsilon_0 > 0$, so Brun–Titchmarsh and partial summation bound its reciprocal prime sum by $O(1/\varphi(q_0h)) \ll_\delta 1/(q_0h)$. Summing the extra weight $1/q_0$ gives $O(1/(h(\log\log X)^2))$. The reciprocal sum over r in its class is

$$\ll_\delta \begin{cases} (\log X)X^{-1/20}, & h > X^{1/20}, \\ h^{-1}, & h \leqslant X^{1/20}, \end{cases}$$

which proves the estimate. The same progression bounds without $\mathfrak{f}$ and (4.19) yield

$$\sum_{q,r}\left(\frac{1}{qr} + \frac{\log\log X}{\log Y}\frac{\mathfrak{f}(qr)}{qr}\right) \ll_\delta \begin{cases} \dfrac{(\log X)(\log\log X)}{hX^{1/20}\log Y}, & X^{1/20} < h \leqslant X^{10/33}, \\ h^{-2}, & h \leqslant X^{1/20}. \end{cases} \tag{4.21}$$

Using (4.13), the contribution for fixed n', h to (4.17) is at most

$$\frac{X\log Y}{(\log X)^2}\frac{dh}{n'}\frac{\log X}{d\log Y}\tau(h) \times \text{the applicable bound in (4.21)}.$$

For $X^{1/20} < h \leqslant X^{10/33}$ this is

$$\ll_\delta \frac{X^{19/20}\log\log X}{\log Y}\frac{\tau(h)}{n'}.$$

Now $\sum_{h \mid s(n')} \tau(h) \leqslant \tau(s(n'))^2 = X^{o(1)}$, and (4.14) shows that the total is

$$\frac{X^{19/20+o(1)}}{d} = o\left(\frac{X}{d\log Y}\right).$$

For $h \leqslant X^{1/20}$ the fixed-n', h contribution is

$$\ll_\delta \frac{X}{\log X}\frac{\tau(h)}{hn'}.$$

Finally,

$$\sum_{h|s(n')} \frac{\tau(h)}{h} \leqslant \left(\frac{\sigma(s(n'))}{s(n')}\right)^2 \leqslant B_\delta^2.$$

Summing over h and then using (4.14) gives $O_\delta(X/(d\log Y))$. Together with the large-h and diagonal contributions, this proves (4.12). □

Proof of Theorem 1.2. For each $d \in \mathcal{D}_X$, Cauchy–Schwarz, Lemma 4.7, and Proposition 4.8 give

$$\#s(\mathcal{A}_d(X)) \geqslant \frac{\#\mathcal{A}_d(X)^2}{\sum_u R_d(u)^2} \gg_\delta \frac{X}{d\log Y}.$$

The image sets are disjoint, and hence

$$\#s(\mathcal{A}(X)) \gg_\delta \frac{X}{\log Y} \sum_{d\in\mathcal{D}_X} \frac{1}{d} \gg_\delta X.$$

If $N = s(M)$ with $M \in \mathcal{A}(X)$, then (4.4) and (4.7) give

$$f(N) \leqslant M < \delta N.$$

Moreover, $N = s(M) < C_\delta M \leqslant C_\delta X$. Replacing X by X/C_δ proves the theorem for $0 < \delta \leqslant 1$, and the case $\delta > 1$ then trivially follows from $\delta = 1$. □

5. Upper tails

The moment bound handles representations with large cofactors. For small cofactors, we consider integers N coprime to all small primes. The divisibility properties of $\sigma(n)$ then force the least common multiple of the divisors of n below e to be large. We begin by estimating a weighted sum over large divisors of $\Lambda(y)$.

5.1. **Sums over large divisors of a least common multiple.**

Lemma 5.1. *For $y, H \geqslant 2$, put*

$$S(y,H) := \sum_{\substack{L|\Lambda(y)\\ L>H}} \frac{\tau(L)}{L}.$$

If $E \to \infty$ and

$$P := \left\lfloor E^{\,1+(2+(\log\log\log E)^{-1/2})\log\log E/\log\log\log E} \right\rfloor,$$

then

$$\log S(E, P/E) \leqslant -\frac{\log\log E}{\sqrt{\log\log\log E}} + O\left(\frac{\log\log E}{\log\log\log E}\right). \tag{5.1}$$

If instead $J \to \infty$,

$$W := \sqrt{\frac{J\log J}{2}}, \quad F := \lfloor e^W \rfloor,$$

then, for every fixed $C > 0$,

$$S\left(F, \frac{e^J}{CW}\right) \leqslant \exp\bigl(-W + o(W)\bigr). \tag{5.2}$$

Proof. For $0 < \alpha \leqslant 1/4$, Rankin's inequality and the Euler product for the divisor function give

$$S(y,H) \leqslant H^{-\alpha} \prod_{p\leqslant y} \sum_{a\geqslant 0} \frac{a+1}{p^{a(1-\alpha)}} = H^{-\alpha} \prod_{p\leqslant y} (1-p^{-1+\alpha})^{-2}. \tag{5.3}$$

Mertens' estimate for $\sum_{p\leqslant y} 1/p$, Chebyshev's bound, and partial summation show that, when $z = \alpha \log y \to \infty$,

$$\sum_{p\leqslant y} p^{-1+\alpha} = \log\log y + O\left(\int_0^z \frac{e^u - 1}{u}\, du\right) = \log\log y + O\left(\frac{e^z}{z}\right). \tag{5.4}$$

The higher terms in the logarithm of the Euler product in (5.3) are $O(1)$ uniformly for $\alpha \leqslant 1/4$.

For (5.1), take $\alpha = \log\log\log E/\log E$. Then

$$\alpha \log(P/E) = \left(2 + \frac{1}{\sqrt{\log\log\log E}}\right)\log\log E + o(1),$$

whereas (5.4) gives

$$2\sum_{p\leqslant E} p^{-1+\alpha} = 2\log\log E + O\left(\frac{\log\log E}{\log\log\log E}\right).$$

Substitution in (5.3) proves the first assertion.

For (5.2), take $\alpha = W/J$. Since

$$\alpha \log F = \frac{1}{2}\log J + o(1), \quad \alpha \log\left(\frac{e^J}{CW}\right) = W + o(W),$$

equation (5.4) gives

$$\sum_{p\leqslant F} p^{-1+\alpha} \ll \log W + \frac{\sqrt{J}}{\log J} = o(W).$$

The second assertion follows once more from (5.3). □

5.2. **Congruence conditions for bounded cofactors.** For sufficiently large A, put

$$L(A) := \frac{\log\log\log A}{(\log A)(\log\log A)}.$$

The moduli $K_{e,d}$ define a periodic set whose intersection with G_A has density zero. Its density estimate will also be useful for the questions about $\underline{\mathrm{d}}(G_A)$ in Section 6.

For $A \geqslant 2$ let $\mathcal{K}_A$ be the set of moduli $K_{e,d}$ with $2 \leqslant e \leqslant A$ and $d \geqslant 1$; it is finite by Section 2. Put

$$V_A := \{N \in \mathbb{N} : k \nmid N \text{ for every } k \in \mathcal{K}_A\}.$$

This is a union of residue classes modulo $W_A := \operatorname{lcm}(\mathcal{K}_A)$, so it has a natural density, and $\mathrm{d}(V_A) \geqslant \varphi(W_A)/W_A > 0$. By inclusion and exclusion the density is the rational number

$$\mathrm{d}(V_A) = 1 - \sum_{\emptyset \neq I \subseteq \mathcal{K}_A} \frac{(-1)^{|I|+1}}{\operatorname{lcm}(I)}.$$

Proposition 5.2. *Fix $A \geqslant 2$. All but $o_A(X)$ of the integers $N \leqslant X$ in G_A are divisible by some $k \in \mathcal{K}_A$. Consequently*

$$1 - \underline{\mathrm{d}}(G_A) \geqslant \mathrm{d}(V_A).$$

Proof. Let $N \leqslant X$ lie in $G_A \cap V_A$. If N is a value of F_1 only, then $N = \sigma(d)$ lies in $\sigma(\mathbb{N})$, which has density zero by Lemma 2.4. Otherwise $N = F_e(d)$ with $2 \leqslant e \leqslant A$. Put $n = ed \leqslant AX$ and $k = K_{e,d} \in \mathcal{K}_A$. By (2.3), $N \equiv \sigma(n) \pmod{k}$, and $k \nmid N$ forces $k \nmid \sigma(n)$, hence $W_A \nmid \sigma(n)$. The pair (e, n) determines N, so these N number at most

$$A \cdot \#\{n \leqslant AX : W_A \nmid \sigma(n)\} = o_A(X)$$

by Lemma 2.3. Thus all but $o_A(X)$ of the elements of $V_A \cap [1, X]$ lie outside G_A, which gives the density bound. □

The next estimate bounds the density of V_A.

Corollary 5.3. *As $A \to \infty$,*

$$\Big(\frac{e^{-\gamma}}{2} - o(1)\Big)L(A) \leqslant \mathrm{d}(V_A) \leqslant \Delta(A).$$

Proof. Every prime $p \leqslant A$ lies in $\mathcal{K}_A$: with $e = p$ and $d = 1$ one has $D = \{1\}$, $M = C = g = 1$ and $K_{p,1} = p$. So V_A consists of A-rough integers, and $\mathrm{d}(V_A) \leqslant \Delta(A)$.

For the lower bound let P be the parameter of Lemma 5.1 with $E = A$. Then $\log P = (2 + o(1))(\log A)(\log\log A)/\log\log\log A$, and Mertens' theorem gives

$$\Delta(P) = \Big(\frac{e^{-\gamma}}{2} + o(1)\Big)L(A).$$

It suffices to show that all but a proportion $o(1)$ of the P-rough integers lie in V_A.

Let N be P-rough and divisible by $k = K_{e,d}$ for some $2 \leqslant e \leqslant A$. Then k is P-rough. Since e/g divides k and $e/g \leqslant A < P$, we must have $e/g = 1$, that is, $e \mid M$ and $k = C$. Now $C \geqslant M \geqslant e \geqslant 2$, so $C > P$, and $C \leqslant (e-1)M$ gives $M > P/A$.

Given M, there are at most $\tau(M)$ choices of e, since $e \mid M$, and e and M together determine $D = \{j < e : j \mid M\}$ and hence C. So at most $\tau(M)$ moduli k share a given M, and for each of them the P-rough multiples of C have density $\Delta(P)/C \leqslant \Delta(P)/M$.

Since $M \mid \Lambda(A)$ and $M > P/A$,

$$\mathrm{d}\{N : N \text{ } P\text{-rough},\ N \notin V_A\} \leqslant \Delta(P) \sum_{\substack{M \mid \Lambda(A) \\ M > P/A}} \frac{\tau(M)}{M} = \Delta(P)\, S(A, P/A) = o(\Delta(P))$$

by (5.1). Hence $\mathrm{d}(V_A) \geqslant (1 - o(1))\Delta(P)$. □

5.3. **Fixed thresholds.**

Proof of Theorem 1.3. Fix an integer $j \geqslant 3$ and, for sufficiently large T, put

$$b_j := \frac{j+1}{j-1}, \quad E := \left\lceil T^{b_j} (\log T)^{3/(j-1)} \right\rceil.$$

Throughout the counting argument, j, T, E are fixed before $X \to \infty$. By Proposition 5.2, all but $o_E(X)$ of the integers in $V_E \cap [1, X]$ have no representation with cofactor at most E. Lemma 2.6 bounds the number of integers $N \leqslant X$ with a representation $n \leqslant TN$ and cofactor greater than E by

$$C_j T^{j+1} E^{1-j} X \leqslant \frac{C_j X}{(\log T)^3}.$$

Moreover, V_E consists of E-rough integers. Its nonsquarefree members up to X therefore number at most

$$X \sum_{p > E} \frac{1}{p^2} \ll \frac{X}{E}.$$

All its members are odd, so Lemma 3.1 discards only $o(X/\log\log\log X)$ unrepresented integers. Every integer surviving these deletions is represented, squarefree, and satisfies $f(N) > TN$. Consequently

$$\underline{\mathrm{d}}\{N \in \mathcal{R} : N \text{ squarefree},\ f(N) > TN\} \geqslant \mathrm{d}(V_E) - \frac{C_j}{(\log T)^3} - O(E^{-1}).$$

Corollary 5.3 and

$$L(E) = \left(\frac{1}{b_j} + o(1)\right) L(T)$$

as $T \to \infty$ show that the right-hand side is at least

$$\left(\frac{e^{-\gamma}}{2b_j} - o(1)\right)\frac{\log\log\log T}{(\log T)(\log\log T)}.$$

Here both error terms are $o(L(T))$ for each fixed j. First let $T \to \infty$ for fixed j, and then let $j \to \infty$. Since $b_j \to 1$, this proves (1.2), independently of the finite representability computation. □

5.4. **Growing thresholds.**

Proof of Theorem 1.4. Fix $0 < \varepsilon < 1$ sufficiently small in terms of η. Choose a fixed $\rho > 0$ sufficiently small, depending only on the absolute constants in Lemma 2.5, and put

$$P := \left\lfloor \rho\frac{\log\log X}{\log\log\log X}\right\rfloor, \quad J := \log P, \quad W := \sqrt{\frac{J\log J}{2}}, \quad F := \lfloor e^W \rfloor, \quad t := e^{(1-\varepsilon)W}.$$

In particular $F < P$ for all sufficiently large X. Complete periods modulo $P\#$ and Mertens' theorem give

$$\#\{X/2 < N \leqslant X : \gcd(N, P\#) = 1\} = \frac{X}{2}\Delta(P) + O(P\#) \asymp \frac{X}{\log\log\log X}, \tag{5.5}$$

because $P\# = \exp(O(P)) = X^{o(1)}$ and $\Delta(P) \asymp 1/J$. The nonsquarefree P-rough integers in this interval number at most

$$X\sum_{p>P}\frac{1}{p^2} \ll \frac{X}{P} = o(X\Delta(P)).$$

Write $Y = tX$. Lemma 2.5 and the union bound show that the number of integers $n \leqslant Y$ for which some prime $p \leqslant P$ fails to divide $\sigma(n)$ is at most

$$O(\sqrt{Y}) + O\left(Y\pi(P)\exp\left(-c_1\frac{\log\log Y}{P}\right)\right). \tag{5.6}$$

The first term treats $p = 2$. The chosen range of P is permitted by Lemma 2.5 after decreasing ρ if necessary. Since

$$Ft \leqslant e^{(2-\varepsilon)W} = (\log\log X)^{o(1)}, \quad \exp\left(-c_1\frac{\log\log Y}{P}\right) = (\log\log X)^{-c_1/\rho+o(1)},$$

we may also ensure $c_1/\rho > 3$. Multiplying (5.6) by the at most F possible cofactors for each integer n gives $o(X\Delta(P))$ pairs (N, e).

Consider a remaining P-rough integer N with a representation $N = F_e(n/e)$ satisfying $n \leqslant tN$ and $e \leqslant F$. Again $e = 1$ is impossible. For $e \geqslant 2$, put

$$L := \operatorname{lcm}\big(\{r : r \mid n,\ r < e\}\big), \quad C := \sum_{\substack{r\mid L\\ r<e}}\frac{L}{r}, \quad R := \sigma(n) - N.$$

The divisors of L below e are exactly the divisors of n below e. By Lemma 2.2,

$$R = \frac{n}{L}C.$$

Since every prime $p \leqslant P$ divides $\sigma(n)$ but not N, the integer $R = (n/L)C$ is P-rough. Writing $u = n/L$, both u and C are therefore P-rough. Since $e \leqslant F < P$ and $e \mid n = Lu$, every prime-power divisor of e already divides L. Consequently

$$n = Lu, \quad \gcd(u, P\#) = 1, \quad \gcd(C, P\#) = 1, \quad e \mid L \mid \Lambda(F).$$

As before $C \geqslant L \geqslant 2$, so $C > P$. On the other hand,

$$C \leqslant \sigma(L) \leqslant L\prod_{p\leqslant F}\left(1 - \frac{1}{p}\right)^{-1} \ll L\log F.$$

There is therefore an absolute constant $C_0 > 0$ such that

$$L > \frac{P}{C_0 W}. \tag{5.7}$$

Uniformly for $L \mid \Lambda(F)$,

$$\log L \leqslant \log \Lambda(F) \ll F = o(\log X), \quad \log(P\#) \ll P = o(\log X).$$

It follows that $tX/L \geqslant P\#$ for all sufficiently large X. Complete periods modulo $P\#$ consequently give

$$\#\{u \leqslant tX/L : \gcd(u, P\#) = 1\} \leqslant \frac{2tX\Delta(P)}{L}.$$

For each fixed L, the admissible cofactors satisfy $e \mid L$, so there are at most $\tau(L)$ possibilities. Hence (5.7) and Lemma 5.1 bound the number of all remaining representations with $e \leqslant F$ by

$$2tX\Delta(P) \sum_{\substack{L \mid \Lambda(F) \\ L > P/(C_0 W)}} \frac{\tau(L)}{L} \leqslant X\Delta(P) \exp\bigl(-\varepsilon W + o(W)\bigr) = o(X\Delta(P)).$$

Choose a fixed integer $k \geqslant 2$ so large that $(k+1)\varepsilon > 2$. Lemma 2.6 bounds the number of integers $N \leqslant X$ having a representation with $n \leqslant tN$ and cofactor $e > F$ by

$$C_k t^{k+1} F^{1-k} X.$$

After division by $X\Delta(P)$, this is at most

$$O_k\Bigl(J \exp\bigl((2 - (k+1)\varepsilon + o(1))W\bigr)\Bigr) = o(1).$$

Thus a positive proportion of the squarefree integers in (5.5) has no representation with $n \leqslant tN$. These integers are odd. Lemma 3.1 discards only $o(X/\log\log\log X) = o(X\Delta(P))$ further integers as unrepresented. Consequently a positive proportion still survives, without any use of the finite representability computation. Finally,

$$\log t = \left(\frac{1-\varepsilon}{\sqrt{2}} + o(1)\right) \sqrt{(\log\log\log X)(\log\log\log\log X)}.$$

Choose ε sufficiently small in terms of η. For $X/2 < N \leqslant X$, replacing X by N in these iterated logarithms changes the right-hand side by $o(1)$ relative to its main term. This proves (1.3).

For (1.4), apply the first assertion with a sufficiently small auxiliary value of η, raise the resulting ratios to the power s, and note that $\log\log\log\log X$ is of strictly smaller order of magnitude than the expression above. The factor $c_\eta/\log\log\log X$ in the proportion of surviving integers can therefore be absorbed into the prescribed exponent loss η. □

6. Bounded representations

We now study the representations themselves and their contribution to the distribution of the least representing integer. For $t > 0$ and $N \geqslant 1$, let

$$r_t(N) := \#\{(e,d) \in \mathbb{N}^2 : F_e(d) = N,\ ed \leqslant tN\},$$
$$r_t^*(N) := \#\{(e,d) \in \mathbb{N}^2 : e \geqslant 2,\ F_e(d) = N,\ ed \leqslant tN\}.$$

The function $r_t^*(N)$ counts representations that do not use the full divisor sum. Equivalently, $r_t(N)$ counts the pairs (n, j) for which $N = \sigma_j(n)$ and $n \leqslant tN$. Thus, on the represented set $\mathcal{R}$, one has $r_t(N) > 0$ if and only if $f(N) \leqslant tN$.

Recall the empirical measures ν_X and the normalization $R(X)$ from (1.5). We regard ν_X as measures on $[0,\infty]$ when taking subsequential limits. A limiting mass at ∞ records escape of mass to infinity. *Tightness* on $[0,\infty)$ by definition means

$$\lim_{T\to\infty}\sup_{X\geqslant 1}\nu_X((T,\infty])=0.$$

The associated density question is

$$\lim_{A\to\infty}\limsup_{X\to\infty}\frac{1}{X}\#\{N\leqslant X : r_A(N)=0\}=0. \tag{6.1}$$

The order of the limits matters: A is fixed before letting $X\to\infty$. The next proposition shows that every fixed residue class has many representation pairs. This does not by itself control the number of distinct integers.

6.1. **A first moment in every residue class.** For fixed $Q\geqslant 1$ and a residue ρ modulo Q, put

$$\underline{\lambda}_{Q,\rho}(A):=\liminf_{X\to\infty}\frac{Q}{X}\sum_{\substack{N\leqslant X\\ N\equiv\rho \pmod Q}} r_A(N).$$

Proposition 6.1. *For every fixed $Q\geqslant 1$ and every residue class ρ modulo Q,*

$$\underline{\lambda}_{Q,\rho}(A)\gg_{Q,\rho}\frac{A}{(\log A)^2}$$

as $A\to\infty$. In particular, $\underline{\lambda}_{Q,\rho}(A)$ tends to infinity in every residue class.

Proof. Fix Q and ρ, and set $d_0=\gcd(\rho,Q)$. We first choose a fixed integer h satisfying

$$\gcd(\sigma(h),Q)=d_0. \tag{6.2}$$

For each prime power $\ell^k\| d_0$, Dirichlet's theorem on primes in arithmetic progressions permits the choice of a different auxiliary prime q_ℓ congruent to 1 modulo every prime dividing Q. When $\ell=2$, we also require $q_2\equiv 1 \pmod 4$. Put

$$h_\ell:=q_\ell^{\ell^k-1},\quad h:=\prod_{\ell^k\| d_0}h_\ell,$$

with $h=1$ when $d_0=1$. The lifting-the-exponent formula gives $v_\ell(\sigma(h_\ell))=k$. If r is a prime divisor of Q with $r\neq\ell$, then $\sigma(h_\ell)\equiv\ell^k\not\equiv 0 \pmod r$. Multiplicativity of σ proves (6.2).

The elementary criterion for a linear congruence now gives a unit $u_\rho \pmod Q$ for which

$$\sigma(h)u_\rho\equiv\rho \pmod Q. \tag{6.3}$$

This includes the class $\rho=0$. Prime-power by prime-power, divide by the corresponding prime power in d_0, solve for a unit modulo the remaining power, and combine the solutions by the Chinese remainder theorem.

Dirichlet's theorem also supplies infinitely many primes e satisfying

$$e>\max(2,h,Q),\quad e\equiv -1 \pmod Q.$$

For each such prime e, define

$$\mathcal{S}_e(u_\rho):=\{t\geqslant 1 : t\equiv u_\rho \pmod Q,\ \gcd(t,e\#)=1\}.$$

The Chinese remainder theorem shows that this periodic set has density

$$\delta_e:=\frac{1}{Q}\prod_{\substack{p\leqslant e\\ p\nmid Q}}\left(1-\frac{1}{p}\right).$$

Every prime factor of $t\in\mathcal{S}_e(u_\rho)$ is larger than e.

For $t \in \mathcal{S}_e(u_\rho)$, the divisors of eht below e are exactly the divisors of h. Complementary divisors therefore give

$$F_e(ht) = \sigma(h)H_e(t), \quad H_e(t) := (e+1)\sigma(t) - et = t + (e+1)s(t). \tag{6.4}$$

Since $Q \mid e+1$, equations (6.3) and (6.4) give

$$F_e(ht) \equiv \sigma(h)t \equiv \rho \pmod{Q}.$$

The corresponding representing integer is $n = eht$. Since $H_e(t) \geqslant t$,

$$\frac{n}{F_e(ht)} \leqslant \frac{eh}{\sigma(h)} \leqslant eh. \tag{6.5}$$

We next bound the represented value. Complementary divisors give

$$\frac{H_e(t)}{t} = 1 + (e+1)\sum_{\substack{a \mid t \\ a>1}} \frac{1}{a}.$$

For a fixed integer $a > 1$ with $\gcd(a, e\#) = 1$, the Chinese remainder theorem shows that the set of $t \in \mathcal{S}_e(u_\rho)$ divisible by a has density δ_e/a. The contribution of a divisor a to the normalized average is at most $(e+1)a^{-2}$, since at most T/a integers up to T are divisible by a. Since e is fixed, convergence of $\sum_{a \geqslant 1} a^{-2}$ justifies interchanging the limit and the divisor sum. Therefore

$$\lim_{T\to\infty} \frac{1}{T} \sum_{\substack{t \leqslant T \\ t \in \mathcal{S}_e(u_\rho)}} \frac{H_e(t)}{t} = \delta_e\left(1 + (e+1) \sum_{\substack{a>1 \\ \gcd(a,e\#)=1}} \frac{1}{a^2}\right) \leqslant 3\delta_e.$$

Every integer $a > 1$ coprime to $e\#$ exceeds e. Hence

$$(e+1) \sum_{\substack{a>1 \\ \gcd(a,e\#)=1}} \frac{1}{a^2} \leqslant (e+1)\sum_{a>e} \frac{1}{a^2} \leqslant \frac{e+1}{e} \leqslant \frac{3}{2},$$

which proves the last inequality. Markov's inequality shows that

$$\mathcal{G}_e := \{t \in \mathcal{S}_e(u_\rho) : H_e(t) \leqslant 6t\}$$

has lower density at least $\delta_e/2$.

For a fixed A, restrict to primes e satisfying

$$\max(2, h, Q) < e \leqslant A/h, \quad e \equiv -1 \pmod{Q}.$$

If $t \in \mathcal{G}_e$ and $t \leqslant X/(6\sigma(h))$, then the represented value in (6.4) is at most X, lies in the class ρ, and has a representing integer $n \leqslant AN$ by (6.5). Distinct pairs (e, t) give distinct representing integers. Indeed, if $e < e'$ and $eht = e'ht'$, then $e \mid t'$, contrary to the fact that every prime factor of t' exceeds e'. If $e = e'$, equality of the representing integers forces $t = t'$. Hence these pairs give distinct representation pairs (n, j), even if some represented values coincide. For each eligible prime e, the number of such $t \leqslant X/(6\sigma(h))$ is at least $(\delta_e/2 + o(1))X/(6\sigma(h))$. Multiplying by Q/X and summing over e gives

$$\underline{\lambda}_{Q,\rho}(A) \geqslant \frac{1}{12\sigma(h)} \sum_{\substack{\max(2,h,Q)<e\leqslant A/h \\ e \text{ prime} \\ e\equiv -1 \pmod{Q}}} \prod_{\substack{p \leqslant e \\ p \nmid Q}} \left(1 - \frac{1}{p}\right). \tag{6.6}$$

Mertens' product theorem gives

$$\prod_{\substack{p \leqslant e \\ p \nmid Q}} \left(1 - \frac{1}{p}\right) \sim \frac{Q}{\varphi(Q)} \frac{e^{-\gamma}}{\log e}.$$

For $Z \geqslant 2$, the prime number theorem in arithmetic progressions, followed by partial summation, gives, as $Z \to \infty$,

$$\sum_{\substack{e \leqslant Z \\ e \text{ prime} \\ e \equiv -1 \pmod{Q}}} \frac{1}{\log e} \sim \frac{Z}{\varphi(Q)(\log Z)^2}.$$

The omitted primes below the fixed threshold $\max(2, h, Q)$ do not change this asymptotic. Taking $Z = A/h$ in (6.6) proves the proposition. □

Remark 6.2. Proposition 6.1 counts representation pairs. A small set of represented values could carry many pairs, so the proposition alone gives no bound for $\#\{N \leqslant X : r_A(N) = 0\}$. Thus the proposition does not by itself imply (6.1).

6.2. **Ranges for prime cofactors.** For prime p, the modulus $K_{p,d}$ from Section 2 is divisible by p. This gives an upper bound for the density of $F_p(\mathbb{N})$.

Corollary 6.3. *Let p be prime. Then $F_p(d) \equiv \sigma(pd) \pmod{p}$ for every $d \geqslant 1$, and*

$$\#\big(F_p(\mathbb{N}) \cap [1, X]\big) \leqslant \frac{X}{p} + o_p(X), \quad \text{so} \quad \overline{\mathrm{d}}\big(F_p(\mathbb{N})\big) \leqslant \frac{1}{p}.$$

Proof. Every $j \in D$ satisfies $j < p$, so $p \nmid M$ and $g = 1$. Hence $p \mid K_{p,d}$, and the congruence follows from (2.3). Now let $N = F_p(d) \leqslant X$ with $p \nmid N$. Then $p \nmid \sigma(pd)$, and $pd \leqslant pN \leqslant pX$ because $d \leqslant F_p(d)$. Distinct N have distinct representing integers pd, so these N number at most $\#\{n \leqslant pX : p \nmid \sigma(n)\}$, which is $o_p(X)$ by Lemma 2.3. Every other value of F_p up to X is a multiple of p. □

For $p = 2$ the bound $1/2$ is weaker than Proposition 6.6 below. For every prime p, the upper bound is $1/p$; the error term $o_p(X)$ is asserted only with p fixed.

6.3. **Bounded cofactors.** For a positive integer A, recall $G_A = \bigcup_{e \leqslant A} F_e(\mathbb{N})$ and put

$$P_A := A\Lambda(A).$$

Every $N = F_e(d)$ with $e \leqslant A$ has a representing integer $n = ed \leqslant AN$, because $F_e(d) \geqslant d$. Thus

$$G_A \subseteq \{N : r_A(N) > 0\}.$$

The upper-tail estimate shows that the complement of each G_A contains a set of positive lower density consisting of squarefree integers.

Corollary 6.4. *For every fixed $A \geqslant 1$,*

$$\underline{\mathrm{d}}(\mathbb{N} \setminus G_A) > 0.$$

More precisely, as $A \to \infty$,

$$\underline{\mathrm{d}}\{N : N \text{ squarefree}, \ N \notin G_A\} \geqslant \left(\frac{e^{-\gamma}}{2} - o(1)\right) \frac{\log\log\log A}{(\log A)(\log\log A)}. \tag{6.7}$$

Proof. If $N \in \mathcal{R}$ and $f(N) > AN$, then $N \notin G_A$. The quantitative assertion follows from Theorem 1.3 with $T = A$. For any fixed smaller A, apply that theorem with one sufficiently large fixed $T \geqslant A$. □

Write

$$\delta_A := \Delta(P_A), \quad \eta_A := \overline{\mathrm{d}}\{N : \gcd(N, P_A\#) > 1, \ N \notin G_A\}.$$

For $2 \leqslant e \leqslant A$, every modulus $K_{e,d}$ has a prime factor at most P_A. Indeed, if $e/g > 1$, use a prime factor of e/g; otherwise $K_{e,d} = C$, where $2 \leqslant C \leqslant (e-1)M \leqslant P_A$. There are only finitely many such moduli. Lemma 2.2 and Lemma 2.3 therefore imply that all but $o_A(X)$

values $F_e(d) \leqslant X$ have a prime factor at most P_A: the representing integers satisfy $ed \leqslant AX$, and each contributes at most A values. The case $e = 1$ is covered by Lemma 2.4. Hence

$$\#\{N \leqslant X : \gcd(N, P_A\#) = 1,\ N \in G_A\} = o_A(X).$$

Thus G_A has density zero among the integers coprime to $P_A\#$. Hence the coprime part of $\mathbb{N} \setminus G_A$ has density δ_A. Since the coprime and noncoprime parts are disjoint,

$$1 - \underline{\mathrm{d}}(G_A) = \delta_A + \eta_A. \tag{6.8}$$

Moreover, Mertens' theorem and the prime number theorem, in the form $\psi(A) = A + o(A)$, give $\log P_A = \log A + \psi(A) = A + o(A)$ and hence

$$\delta_A = \frac{e^{-\gamma} + o(1)}{A}. \tag{6.9}$$

The two terms in (6.8) are of different orders, and it is η_A that dominates. Recall the notation

$$L(A) = \frac{\log\log\log A}{(\log A)(\log\log A)}.$$

Proposition 6.5. *As $A \to \infty$,*

$$\eta_A \geqslant \Big(\frac{e^{-\gamma}}{2} - o(1)\Big)L(A),$$

so $A\eta_A \to \infty$.

Proof. By (6.7), $1-\underline{\mathrm{d}}(G_A) \geqslant (e^{-\gamma}/2-o(1))L(A)$. Since $AL(A) \to \infty$, (6.9) gives $\delta_A = o(L(A))$. Subtracting in (6.8) proves the first claim, and the second follows from $AL(A) \to \infty$. □

One might guess that coprimality to $P_A\#$ is asymptotically the only obstruction, so that $1 - \underline{\mathrm{d}}(G_A) \sim e^{-\gamma}/A$. The proposition rules this out. The integers missed by G_A for other reasons outnumber the coprime ones by a factor that tends to infinity.

Now set

$$\theta_2 := \overline{\mathrm{d}}\big(F_2(\mathbb{N})\big).$$

Proposition 6.6. *One has*

$$\theta_2 \leqslant \frac{1}{2} - 0.0602757 = 0.4397243.$$

Proof. The only divisor of $2d$ exceeding d is $2d$, and hence

$$F_2(d) = \sigma(2d) - 2d = s(2d). \tag{6.10}$$

Thus $F_2(\mathbb{N}) \subseteq s(\mathbb{N})$ and the cofactor-two range omits every untouchable number. Write

$$\mathcal{U} := \mathbb{N} \setminus s(\mathbb{N}).$$

We first note that the odd part of $F_2(\mathbb{N})$ is negligible. If n is even, then $s(n)$ is odd precisely when $\sigma(n)$ is odd, which occurs precisely when n is a square or twice a square. For even $n \geqslant 4$ one also has $s(n) \geqslant 1 + n/2$; the exceptional case $n = 2$ is harmless. Thus $s(n) \leqslant X$ implies $n < 2X$, apart from at most this one value. Consequently,

$$\#\{N \leqslant X : N \text{ odd},\ N \in F_2(\mathbb{N})\} = O(\sqrt{X}). \tag{6.11}$$

Almost all untouchable numbers are even. Indeed, let N be odd. If $N - 1 = p + q$ with distinct primes p and q, then the proper divisors of pq are $1, p, q$, and therefore $s(pq) = N$. The exceptional-set theorem of Montgomery and Vaughan shows that the even integers up to X that are not a sum of two primes number $O(X^{1-\kappa})$ for some $\kappa > 0$; the diagonal possibilities $N - 1 = 2p$ contribute $O(X/\log X)$ more [13]. Hence

$$\#\{N \leqslant X : N \text{ odd},\ N \in \mathcal{U}\} = o(X). \tag{6.12}$$

Chen and Zhao proved [3] that

$$\underline{\mathrm{d}}(\mathcal{U}) \geqslant 0.0602757.$$

By (6.12), the number of even untouchables not exceeding X is at least $(0.0602757 + o(1))X$. There are $X/2 + O(1)$ even integers up to X, and by (6.11) all but $O(\sqrt{X})$ elements of $F_2(\mathbb{N}) \cap [1, X]$ are even. Since (6.10) shows that this range misses every member of $\mathcal{U}$, we obtain

$$\#\big(F_2(\mathbb{N}) \cap [1, X]\big) \leqslant \frac{X}{2} - 0.0602757X + o(X).$$

Taking the upper limit proves the proposition. □

Corollary 6.7. *For η_2 defined above,*

$$\eta_2 \geqslant \frac{2}{3} - 0.4397243 > 0.2269.$$

Proof. Since $F_1(d) = \sigma(d)$,

$$G_2 = \sigma(\mathbb{N}) \cup F_2(\mathbb{N}).$$

Lemma 2.4 implies that $\sigma(\mathbb{N})$ has density zero, and therefore $\overline{\mathrm{d}}(G_2) \leqslant \theta_2$. Moreover, $\Lambda(2) = 2$, $P_2 = 4$, $P_2\# = 6$, and so $\delta_2 = 1/3$. Using (6.8),

$$\eta_2 = 1 - \underline{\mathrm{d}}(G_2) - \delta_2 \geqslant \frac{2}{3} - \overline{\mathrm{d}}(G_2) \geqslant \frac{2}{3} - 0.4397243 > 0.2269.$$ □

We conjecture that bounded cofactors cover sets of lower density tending to one as the cofactor bound grows.

Conjecture 6.8. *As $A \to \infty$ through the positive integers,*

$$\underline{\mathrm{d}}(G_A) \longrightarrow 1.$$

Equivalently, $\eta_A = o(1)$.

The equivalence in Conjecture 6.8 follows from (6.8) and (6.9).

Corollary 5.3 shows where the congruence argument stops. The set V_A consists of A-rough integers, so the congruence conditions say nothing about integers with a prime factor at most A that are missed by G_A. The A-rough part of $\mathbb{N} \setminus G_A$ has upper density at most $\Delta(A)$, which tends to zero. Conjecture 6.8 is therefore equivalent to the assertion that the part of $\mathbb{N} \setminus G_A$ with a prime factor at most A has upper density tending to zero. Every element of that part satisfies some congruence condition and nevertheless lies outside G_A.

The conjecture that $\underline{\mathrm{d}}(G_E) \to 1$ as $E \to \infty$ is equivalent to tightness of the empirical measures ν_X.

Proposition 6.9. *The empirical measures ν_X form a tight family on $[0, \infty)$ if and only if*

$$\lim_{E \to \infty} \overline{\mathrm{d}}(\mathbb{N} \setminus G_E) = 0. \tag{6.13}$$

Equivalently,

$$\lim_{E \to \infty} \underline{\mathrm{d}}(G_E) = 1.$$

Thus Conjecture 6.8 is exactly the assertion that the family $\{\nu_X\}$ is tight.

Proof. The summand d occurs in $F_e(d)$, so $F_e(d) \geqslant d$. Hence

$$N \in G_E \quad \Longrightarrow \quad f(N) \leqslant ed \leqslant EN \quad \text{for some } e \leqslant E.$$

Using (1.1) to replace the normalization by $X + O(1)$, we obtain

$$\limsup_{X \to \infty} \nu_X((E, \infty]) \leqslant \overline{\mathrm{d}}(\mathbb{N} \setminus G_E).$$

Thus (6.13) makes the tail uniformly small for all sufficiently large X. For X in a bounded interval, only finitely many represented integers occur, so their ratios have a common finite upper bound. This proves tightness for the whole family.

Conversely, fix $T \geqslant 1$ and an integer $k \geqslant 2$. Every represented $N \notin G_E$ with $f(N) \leqslant TN$ has a representation with cofactor $e > E$. Lemma 2.6 therefore gives

$$\overline{\mathrm{d}}(\mathbb{N} \setminus G_E) \leqslant \limsup_{X\to\infty} \nu_X((T,\infty]) + C_k T^{k+1} E^{1-k}.$$

If the family is tight, first let $E \to \infty$ and then let $T \to \infty$. The right-hand side tends to zero. Finally, $\overline{\mathrm{d}}(\mathbb{N} \setminus G_E) = 1 - \underline{\mathrm{d}}(G_E)$ identically. □

Since $r_A(N) > 0$ is equivalent, apart from the two unrepresented integers, to $f(N) \leqslant AN$, Proposition 6.9 also shows that the density-one statement (6.1) is equivalent to Conjecture 6.8.

7. Counting measures and their convergence

For each integer $e \geqslant 1$, we represent the residue classes modulo $\Lambda(e)$ by the integers a with $0 \leqslant a < \Lambda(e)$, and put

$$c_e(a) := \sum_{\substack{1\leqslant j<e \\ j\mid a}} \frac{1}{j}.$$

Since every integer j with $1 \leqslant j \leqslant e$ divides $\Lambda(e)$, if $n \equiv a \pmod{\Lambda(e)}$, then $j \mid n$ if and only if $j \mid a$ for $1 \leqslant j \leqslant e$. Thus both $c_e(a)$ and the condition $e \mid a$ depend only on the residue class of a. Write $\mathcal{D}$ for the limiting distribution of $h(n) := \sigma(n)/n$. Davenport's theorem [4] and its version for arithmetic progressions by Pollack [15, Lemma 2] give an atomless measure $\nu_{a,e}$ characterized by

$$\nu_{a,e}(B) := \lim_{Y\to\infty} \frac{1}{Y} \#\{n \leqslant Y : n \equiv a \pmod{\Lambda(e)},\ h(n) \in B\}$$

for every continuity set B. In particular, each $\nu_{a,e}$ has total mass $1/\Lambda(e)$, and

$$\sum_{a=0}^{\Lambda(e)-1} \nu_{a,e} = \mathcal{D}, \quad 0 \leqslant \nu_{a,e} \leqslant \mathcal{D}. \tag{7.1}$$

The measures $\nu_{a,e}$ yield explicit continuous limits for the mean numbers of bounded representations.

Proposition 7.1. *For each $t > 0$ the limits*

$$W(t) := \lim_{X\to\infty} \frac{1}{X} \sum_{N\leqslant X} r_t(N), \quad W^*(t) := \lim_{X\to\infty} \frac{1}{X} \sum_{N\leqslant X} r_t^*(N)$$

exist, are finite, and depend continuously on t. More explicitly,

$$W(t) = \sum_{e\geqslant 1} w_e(t), \quad W^*(t) = \sum_{e\geqslant 2} w_e(t),$$

where

$$w_e(t) := \sum_{\substack{0\leqslant a<\Lambda(e) \\ e\mid a}} \int_{h-c_e(a)\geqslant 1/t} \frac{d\nu_{a,e}(h)}{h - c_e(a)}.$$

Both series converge uniformly for $0 < t \leqslant T$, for every fixed $T > 0$, and $W(t) \to 0$ as $t \downarrow 0$.

Proof. For $n \equiv a \pmod{\Lambda(e)}$ with $e \mid a$, complementary divisors give the exact identities

$$F_e(n/e) = n g_e(n), \quad g_e(n) = h(n) - c_e(a).$$

In particular $g_e(n) \geqslant 1/e$. A representation $N = F_e(n/e)$ with $N \leqslant X$ and $n/N \leqslant t$ is characterized by

$$g_e(n) \geqslant 1/t, \quad n g_e(n) \leqslant X.$$

For completeness, the progression theorem also yields the joint limit

$$\frac{1}{X}\sum_{\substack{n\leqslant tX\\ n\equiv a \pmod{\Lambda(e)}}} \delta_{(n/X,h(n))} \implies \mathbf{1}_{[0,t]}(u)\,du\,d\nu_{a,e}(h).$$

Indeed, on a rectangle $(\alpha,\beta]\times B$ the assertion follows by subtracting the progression counts at βX and αX. The limiting measure in the variable h is tight, so rectangle approximation proves the joint weak convergence. The boundary of $\{h-c_e(a)\geqslant 1/t,\ u(h-c_e(a))\leqslant 1\}$ has zero product measure: the horizontal boundary has zero measure by atomlessness, and for each $h>c_e(a)$ the curved boundary contains only one value of u. The support condition $h-c_e(a)\geqslant 1/e$ holds for $\nu_{a,e}$ because it holds for every integer n in the progression. We may therefore pass to the limit in the two inequalities defining the representations. Tonelli–Fubini's theorem gives

$$\begin{aligned}\lim_{X\to\infty}\frac{1}{X}\#\{n\equiv a \pmod{\Lambda(e)} : g_e(n)\geqslant 1/t,\ ng_e(n)\leqslant X\}&\\ =\int_0^t \nu_{a,e}\{h : h-c_e(a)\geqslant 1/t,\ h-c_e(a)\leqslant 1/u\}\,du&\\ =\int_{h-c_e(a)\geqslant 1/t}\frac{d\nu_{a,e}(h)}{h-c_e(a)}.&\end{aligned}$$

For the cofactor tail, Lemma 2.6 gives, for every integer $k\geqslant 2$ and every $E\geqslant 1$,

$$\begin{aligned}\frac{1}{X}\sum_{N\leqslant X}\#\{(e,d) : e>E,\ F_e(d)=N,\ ed\leqslant tN\}&\leqslant \frac{t^k}{X}\sum_{n\leqslant tX}\sum_{\substack{e\mid n\\ e>E}} g_e(n)^k\\ &\leqslant C_k t^{k+1}E^{1-k}.\end{aligned}$$

This proves both limits and uniform convergence on $0<t\leqslant T$. Each fixed $w_e(t)$ is continuous by atomlessness, and tends to zero with t since its integrand is bounded by t on its domain. The uniform tail proves all remaining assertions. □

Proposition 6.1 with $Q=1$ also shows that $W(t)\gg t/(\log t)^2$ as $t\to\infty$. Since $w_1(t)\leqslant 1$, the same lower bound holds for $W^*(t)$.

7.1. **Singularity and continuity of subsequential limits.** All subsequential limits of the empirical ratio distributions have atomless, nonzero finite parts carried by a single Lebesgue-null set.

Theorem 7.2. *There is a fixed σ-compact, Lebesgue-null set $\mathcal{Z}\subset(0,\infty)$ such that every weak subsequential limit ν of ν_X on $[0,\infty]$ satisfies*

$$0\in\operatorname{supp}(\nu),\quad \nu(\{0\})=0,$$
$$\nu\big((0,\infty)\setminus\mathcal{Z}\big)=0,\quad \nu(\{x\})=0\quad(0<x<\infty).$$

Hence its restriction to $[0,\infty)$ is a nonzero singular continuous measure. If (6.13) holds, every subsequential limit is a singular continuous probability measure on $[0,\infty)$; in particular, none admits a Lebesgue density. Neither uniqueness of the limit nor tightness is asserted unconditionally.

Proof. Erdős's singularity theorem asserts that the limiting distribution $\mathcal{D}$ of $\sigma(n)/n$ is singular with respect to Lebesgue measure; see the account in [6, p. 59]. By regularity there is a σ-compact Lebesgue-null set $K\subset[1,\infty)$ with $\mathcal{D}(K)=1$. For $c\in\mathbb{R}$, write

$$\psi_c(h):=\frac{1}{h-c}$$

for $h > c$, and define the positive measure

$$\mathcal{W} := \sum_{e\geqslant 1} \sum_{\substack{0\leqslant a<\Lambda(e)\\ e\mid a}} (\psi_{c_e(a)})_*\big(\mathbf{1}_{\{h>c_e(a)\}}\psi_{c_e(a)}(h)\, d\nu_{a,e}(h)\big).$$

Proposition 7.1 gives

$$\mathcal{W}((0,t]) = W(t) < \infty$$

for $t > 0$, so $\mathcal{W}$ is locally finite and atomless. By (7.1), every $\nu_{a,e}$ is carried by K. Therefore $\mathcal{W}$ is carried by

$$\mathcal{Z} := \bigcup_{e\geqslant 1} \bigcup_{\substack{0\leqslant a<\Lambda(e)\\ e\mid a}} \bigcup_{j\geqslant 1} \psi_{c_e(a)}\big(K \cap [c_e(a) + 1/j, j]\big).$$

Each map in the last display is Lipschitz on its indicated set. Thus $\mathcal{Z}$ is σ-compact and Lebesgue null.

For each X, define a measure that counts all representations:

$$\mathcal{W}_X := \frac{1}{X} \sum_{N\leqslant X} \sum_{\substack{e,d\geqslant 1\\ F_e(d)=N}} \delta_{ed/N}.$$

On every bounded ratio interval it is finite. Proposition 7.1 and continuity of W show that $\mathcal{W}_X \to \mathcal{W}$ vaguely on $(0,\infty)$. For every nonnegative $\varphi \in C_c((0,\infty))$, choosing a representation with $ed = f(N)$ for each represented integer N gives

$$\int \varphi\, d\nu_X \leqslant \frac{X}{R(X)} \int \varphi\, d\mathcal{W}_X.$$

Along any weakly convergent subsequence $\nu_{X_j} \Rightarrow \nu$, pass to the limit to obtain

$$\int \varphi\, d\nu \leqslant \int \varphi\, d\mathcal{W}.$$

Hence $\nu|_{(0,\infty)} \leqslant \mathcal{W}$. This proves both singularity on the common null set $\mathcal{Z}$ and the absence of positive finite atoms.

Finally, the same comparison of measures and the portmanteau theorem give, for every $t > 0$,

$$\nu(\{0\}) \leqslant \nu([0,t)) \leqslant \liminf_{j\to\infty} \nu_{X_j}([0,t)) \leqslant W(t).$$

Let $t \downarrow 0$ and use Proposition 7.1. To see that zero nevertheless belongs to the support, fix $\delta > 0$. Theorem 1.2 and the closed-set direction of portmanteau give

$$\nu([0,\delta]) \geqslant \limsup_{j\to\infty} \nu_{X_j}([0,\delta]) > 0.$$

Applying this with arbitrarily small δ proves the support claim and shows that the finite part is nonzero. Finally, tightness excludes mass at ∞ in every weak subsequential limit on $[0,\infty]$, so Proposition 6.9 gives the last assertion. □

The tail estimates imply divergence of positive and logarithmic moments, while inverse moments remain finite and pass to subsequential limits.

Corollary 7.3. *Every weak subsequential limit ν of ν_X on $[0,\infty]$ satisfies, as $T \to \infty$,*

$$\nu((T,\infty]) \geqslant \left(\frac{e^{-\gamma}}{2} - o(1)\right) \frac{\log\log\log T}{(\log T)(\log\log T)}. \tag{7.2}$$

Moreover, for every $s > 0$,

$$\lim_{X\to\infty} \int x^s\, d\nu_X(x) = \infty, \qquad \int_{[0,\infty]} x^s\, d\nu(x) = \infty.$$

The logarithmic means also satisfy

$$\lim_{X\to\infty}\int \log x\, d\nu_X(x)=\infty, \qquad \int_{[0,\infty]} \log x\, d\nu(x)=\infty.$$

In contrast, for every $s>0$,

$$\sup_{X\geqslant 1}\int x^{-s}\, d\nu_X(x)<\infty, \qquad \int x^{-s}\, d\nu(x)<\infty,$$

where ∞^{-s} is interpreted as 0. Along every weakly convergent subsequence, the inverse moments converge to those of the limiting measure. In particular, the geometric means of $f(N)/N$ tend to infinity; even assuming tightness, every subsequential limit that is a probability measure on $[0,\infty)$ has infinite logarithmic mean and infinite moments of all positive orders.

Proof. The closed-set inequality in the portmanteau theorem and Theorem 1.3 give

$$\nu([T,\infty])\geqslant \limsup_{j\to\infty}\nu_{X_j}([T,\infty])\geqslant \liminf_{X\to\infty}\nu_X((T,\infty]).$$

Theorem 7.2 excludes an atom at T, proving (7.2). For either ν_X or ν, the sth moment is at least T^s times its mass on $(T,\infty]$. First take the lower limit in X when appropriate, and then let $T\to\infty$ in the preceding lower bound.

For the logarithmic means, write $\log^+ x=(\log x)_+$ and $\log^- x=(-\log x)_+$. For a fixed sufficiently large T_0, layer cake and Fatou's lemma give

$$\begin{aligned}\liminf_{X\to\infty}\int \log^+ x\, d\nu_X(x)&\geqslant \int_{T_0}^{B}\liminf_{X\to\infty}\nu_X((T,\infty])\,\frac{dT}{T}\\&\geqslant \left(\frac{e^{-\gamma}}{4}-o(1)\right)(\log\log\log B)^2\end{aligned}$$

as $B\to\infty$. The same calculation using (7.2) shows $\int \log^+ x\, d\nu(x)=\infty$. On the other hand, Theorem 1.1 and the bound $X/R(X)\leqslant 3$ give, for some $c>0$ and all sufficiently large u,

$$\sup_X \nu_X([0,e^{-u}])\ll \exp\bigl(-\exp(e^{cu})\bigr).$$

Layer cake therefore bounds $\int \log^- x\, d\nu_X(x)$ uniformly in X. To pass the bound to ν, apply weak convergence to the continuous bounded functions $\min(M,\log^- x)$ on $[0,\infty]$, defined to equal M at zero and zero at infinity, and then let $M\to\infty$. Subtracting the bounded negative parts proves both logarithmic assertions.

Finally, the same uniform lower-tail bound and integration of distribution functions show that

$$\sup_{X\geqslant 1}\int_{\{x<1\}} x^{-s}\, d\nu_X(x)\leqslant 1+s\int_0^\infty e^{su}\sup_{X\geqslant 1}\nu_X([0,e^{-u}])\,du<\infty.$$

Apply this also with $2s$ in place of s to obtain uniform integrability of x^{-s}. The bounded truncations $\min(M,x^{-s})$ are continuous on the compactified half-line when assigned their limiting endpoint values. Weak convergence, followed by removal of the truncation, proves the claimed convergence and finiteness of inverse moments. □

7.2. **Multiple representations and convergence.** For each N, the quantity $(r_t^*(N)-1)_+$ counts representations with $e\geqslant 2$ and $ed\leqslant tN$ beyond the first, if any. The following result relates convergence of their average to convergence of ν_X.

Proposition 7.4. *Put*

$$K_X(t):=\frac{1}{X}\sum_{N\leqslant X}(r_t^*(N)-1)_+.$$

For every fixed $t>0$,

$$\nu_X([0,t])=W^*(t)-K_X(t)+o(1). \tag{7.3}$$

Consequently, $\lim_{X\to\infty}\nu_X([0,t])$ *exists if and only if* $K_X(t)$ *converges. The whole sequence* ν_X *converges weakly on* $[0,\infty]$ *if and only if* $K_X(t)$ *converges for every positive rational* t*.*

Proof. If $f(N)\leqslant tN$ but $r_t^*(N)=0$, every representation with $ed\leqslant tN$ has cofactor $e=1$; hence N belongs to range(σ). Lemma 2.4 therefore shows that

$$\nu_X([0,t])=\frac{1}{X}\#\{N\leqslant X : r_t^*(N)>0\}+o(1).$$

For every nonnegative integer r, $\mathbf{1}_{\{r>0\}}=r-(r-1)_+$. Sum this identity over $N\leqslant X$ with $r=r_t^*(N)$ and apply Proposition 7.1.

For the final assertion, all weak subsequential limits on $[0,\infty]$ have no finite atoms by Theorem 7.2. If the quantity $K_X(t)$ converges for each positive rational t, the displayed identity forces any two subsequential limits to agree on $[0,t]$ at all those thresholds, hence everywhere, including at infinity. Compactness then gives convergence of the full sequence. Conversely, weak convergence implies convergence of the mass of each $[0,t]$, because its boundary has zero limiting measure, and the identity gives convergence of $K_X(t)$. □

The next result shows that $K_X(t)$ is bounded away from zero for $t\geqslant 2$ and sufficiently large X.

Proposition 7.5. *For every* $t\geqslant 2$*,*

$$\liminf_{X\to\infty}K_X(t)>\frac{1}{9}.$$

Proof. Put $Q=5\#=30$, so that $\Delta(5)=\varphi(Q)/Q=4/15$, and let

$$\mathcal{C}:=\{2,4,8,10\}.$$

For $D\in\mathcal{C}$ and $\gcd(u,Q)=1$, consider the representation with cofactor 2

$$N:=s(Du)=\sigma(D)\sigma(u)-Du=u\big(s(D)+\sigma(D)(h(u)-1)\big). \tag{7.4}$$

The representing integer $n=Du$ is even, so $N=s(n)\geqslant n/2$ and $n/N\leqslant 2\leqslant t$.

Let μ_Q be the limiting measure for $h(u)$ over integers u coprime to Q, with counts up to Y normalized by $1/Y$. Its total mass is $\Delta(5)$. Expanding $h(u)$ as a divisor sum gives

$$\frac{1}{\Delta(5)}\int(h-1)\,d\mu_Q(h)=\sum_{\substack{d\geqslant 1\\ \gcd(d,Q)=1}}\frac{1}{d^2}-1=\frac{8\pi^2}{75}-1=:H.$$

The passage to this first moment is justified by the uniformly bounded second moment of $h(u)$, which follows from Lemma 4.1 with $q=3$. The joint weak convergence argument in Proposition 7.1 yields

$$\begin{aligned}W_D&:=\lim_{X\to\infty}\frac{1}{X}\#\{u:\gcd(u,Q)=1,\ s(Du)\leqslant X\}\\&=\int\frac{d\mu_Q(h)}{s(D)+\sigma(D)(h-1)}\geqslant\frac{\Delta(5)}{s(D)+\sigma(D)H},\end{aligned}$$

where the last inequality is Jensen's inequality.

Choose the fixed modulus $q=2^4\cdot 3\cdot 5^2$. Lemma 2.3 implies that, among the integers $n=Du\leqslant 2X$ under consideration, all but $o(X)$ satisfy

$$v_p(\sigma(n))>v_p(D)=v_p(n)$$

for $p=2,3,5$. For every remaining represented value N in (7.4),

$$v_p(N)=v_p(\sigma(n)-n)=v_p(D),$$

again for $p=2,3,5$. Thus, up to $o(X)$ exceptions, these values N belong to the periodic set

$$\mathcal{T}_D:=\{Dv:\gcd(v,30)=1\},\quad \mathrm{d}(\mathcal{T}_D)=\frac{\Delta(5)}{D}.$$

If $\rho_D(N)$ denotes the number of parameters u giving N in (7.4), it follows that

$$\liminf_{X\to\infty}\frac{1}{X}\sum_{N\leqslant X}(\rho_D(N)-1)_+ \geqslant \Delta(5)\left(\frac{1}{s(D)+\sigma(D)H}-\frac{1}{D}\right).$$

The four integers $D\in\mathcal{C}$ have distinct valuation vectors at $2,3,5$, whereas u is coprime to 30. Thus the sets of integers Du, and hence the representations with cofactor 2, are disjoint. The elementary inequality

$$\sum_{D\in\mathcal{C}}(\rho_D(N)-1)_+ \leqslant \left(\sum_{D\in\mathcal{C}}\rho_D(N)-1\right)_+ \leqslant (r_t^*(N)-1)_+$$

therefore allows the four lower bounds to be added.

Finally, $\pi<22/7$ gives $H<197/3675$. Since

$$(D,s(D),\sigma(D))\in\{(2,1,3),(4,3,7),(8,7,15),(10,8,18)\},$$

all four terms are positive and exact rational arithmetic gives

$$\begin{aligned}\liminf_{X\to\infty}K_X(t) &> \frac{4}{15}\sum_{D\in\mathcal{C}}\left(\frac{1}{s(D)+197\sigma(D)/3675}-\frac{1}{D}\right)\\ &= \frac{3492203725711}{31001587618275}>\frac{1}{9}.\end{aligned}$$

This lower bound proves that the term $K_X(t)$ in (7.3) is substantial; it does not prove that the term converges. □

8. Further questions

The bounds on the two tails and the structural results above leave several questions about the typical size of the least representing integer.

Theorem 1.3 gives a fixed-level lower bound of order

$$\frac{\log\log\log T}{(\log T)(\log\log T)},$$

while Theorem 1.4, for each fixed $\eta>0$, reaches

$$\exp\left(\left(\frac{1}{\sqrt{2}}-\eta\right)\sqrt{(\log\log\log N)(\log\log\log\log N)}\right)$$

on $\gg_\eta X/\log\log\log X$ integers. What are the correct fixed-level tail and the largest growing scale with a comparable counting estimate? In particular, does the constant $1/\sqrt{2}$ reflect the problem itself or the present estimate for the sum over large divisors of $\Lambda(y)$? At the lower tail, it also remains to quantify the positive-density result as the ratio tends to zero.

Proposition 6.9 identifies tightness with $\underline{\mathrm{d}}(G_E)\to 1$. The periodic set V_E lies among the E-rough integers, whose density tends to zero. Tightness is therefore equivalent to the upper density tending to zero of those integers outside G_E that have a prime factor at most E. Each such integer N satisfies at least one of the divisibility conditions $k\mid N$ with $k\in\mathcal{K}_E$, but is still not a value of any F_e with $e\leqslant E$. The congruence arguments here do not control this residual set.

The limiting first moments $W(t)$ and $W^*(t)$ exist, and Proposition 6.1 shows that the first moment of bounded representations grows in every fixed residue class. Nevertheless, the lower density of integers with a representation satisfying $ed\leqslant tN$ depends on how many such representations give the same integer. Proposition 7.4 gives an exact criterion at each threshold: convergence of the empirical distribution function is equivalent to convergence of $K_X(t)$. The lower bound $\liminf K_X(t)>1/9$ for $t\geqslant 2$ confirms that additional representations of the same integer cannot be neglected. It provides no convergence result. A second-moment bound comparable to the square of the first moment could give a positive lower bound for

this density. To prove that this lower density tends to one as $t \to \infty$ would require an additional argument, for example a sufficiently small normalized variance for the number of representations in a suitable family.

Every weak subsequential limit of ν_X on $[0,\infty]$ has a singular continuous restriction to $[0,\infty)$ carried by the same null set, with zero in its support and no atom there. If tightness holds, the resulting probability measures have unbounded support, infinite logarithmic mean, and infinite moments of every positive order. It remains to prove or disprove tightness, convergence of the full sequence, uniqueness of its subsequential limits, and full support on $(0,\infty)$. The divergence of arithmetic and geometric means is compatible with tightness and answers none of these existence or uniqueness questions.

We have $\mathrm{d}(F_1(\mathbb{N})) = 0$, $\overline{\mathrm{d}}(F_2(\mathbb{N})) \leqslant 0.4397243$, and $\overline{\mathrm{d}}(F_p(\mathbb{N})) \leqslant 1/p$ for every prime p. Luca and Pomerance's work gives positive lower density for $F_2(\mathbb{N})$ [12]. It would be useful to decide which other sets $F_e(\mathbb{N})$, with e fixed, have positive lower density, whether their natural densities exist, and how their overlaps affect the density of G_E.

Author contributions

All authors contributed equally to this work.

Methodology and AI usage

This paper presents the results of a long-term large-scale collaborative project, lasting from November 2025 to August 2026. Its partial results were gradually announced and presented on the forum of the *Erdős Problems* website [2], which presents a reliable timeline of the project's development. The authors contributed their thoughts, ideas, proofs, and formalizations, some of which were assisted by generative AI tools. More specifically,

- Principia Math's autonomous research harness, using the models OpenAI's GPT-5.5 and Anthropic's Claude Opus 4.8, resolved the limsup question in Erdős Problem 1054, proving the lim sup variant of Theorem 1.3.
- Section 3 was initially redacted and the corresponding code was verified by an internal OpenAI model. OpenAI's GPT-5.6 Sol was used in the first formalization of the corresponding argument.
- The proofs of Lemmas 4.6 and 4.7, Proposition 4.8, Lemma 2.2, Proposition 5.2, and Corollary 5.3 were obtained with the help of Anthropic's Claude Opus 5 and audited by Claude Fable 5 and GPT-6 Astra. Opus helped derive the sieve estimates transferred from [12] and suggested the congruence behind Lemma 2.2.
- The proofs of Proposition 6.1, the complement identity (6.8) with the lower bound for η_A, Proposition 6.6 with its corollary, and the upper bound $\overline{\mathrm{d}}(F_p(\mathbb{N})) \leqslant 1/p$ were generated with the help of Anthropic's Claude Opus 5 and OpenAI's GPT-5.6 Sol, and audited by Claude Fable 5 and GPT-6 Astra. The same models were used for polishing the writeup of the corresponding parts and for numerical checks.

The authors independently verified the AI-assisted proofs, carefully inspected the AI-generated parts of the text, rewrote them as appropriate, and they assume full responsibility for the final manuscript. Furthermore, OpenAI's GPT-6 Astra was used for several rounds of copyediting and proofreading.

Formalization

The results of this paper have been formalized in the Lean 4 proof assistant with the Mathlib library. The formalization is available at

`https://github.com/PrincipiaMathOrg/math-solutions/tree/main/erdos1054`

together with a list matching each numbered result of the paper to its Lean statement.

Of the paper's 37 numbered results, 33 are proved in Lean without any additional hypotheses, including Theorems 1.1, 1.2, 1.3 and 1.4. These proofs rely only on Lean's standard axioms, and an independent checker (Comparator) confirms that each of them establishes exactly the stated result.

The remaining four results, namely Lemma 3.6, Proposition 3.7, Theorem 3.2 and its equivalent form (1.1), rest on Helfgott's ternary Goldbach theorem [11]. For these, Helfgott's argument itself is formalized, down to a list of explicitly stated inputs from the literature: published machine computations, namely those of Helfgott and of Platt, and published theorems such as the explicit prime estimates of Rosser and Schoenfeld [16]. These inputs are taken as hypotheses rather than proved in Lean, so the four results are verified conditionally on them.

The first formalization of Theorem 3.2 is due to H.C., who formalized an argument by J.F. (`https://github.com/hs-chae/erdos1054_hyunsik`). His development is included in the repository, both in its original form and as part of the main formalization, where the hypotheses it assumes are derived from the same cited inputs. Theorem 3.2 therefore has two independent formal proofs.

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