COFINITE ZEROS OF HIGH DERIVATIVES

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Abstract

We construct a transcendental entire function f for which every nonempty open subset of the complex plane contains a zero of $f^{(n)}$ for all sufficiently large n. Earlier Gaussian proposals by Almeida and Chojecki use expected-zero and Offord-type estimates. Here the coefficients are independent and uniformly distributed on the closed unit disk. A saddle-point estimate, one-coordinate anti-concentration and the mean value property of $\log |f^{(n)}|$ give summable fixed-disk hole probabilities. The selected function has order exactly two and satisfies $|f(z)| \leq \sqrt{2}\exp(|z|^2)$. It is therefore a counterexample to Boas and Reddy's printed assertion that every transcendental entire function of order at most two and finite type admits an arbitrarily large fixed disk zero-free for infinitely many successive derivatives.

2020 *Mathematics subject classification:* primary 30D20; secondary 30B20, 30C15, 60F15.

Keywords and phrases: entire function, successive derivatives, zero sets, random analytic function, hole probability.

1. Introduction

Erdős asked whether there is a nonzero entire function f such that, for every strictly increasing sequence $(n_k)_{k\geq 0}$ of nonnegative integers, the union of the zero sets of $f^{(n_k)}$ is dense in the complex plane [11, p. 72]. On the same page he wrote that the existence of both displayed functions had been proved "more than ten years ago"; the attached note lists four papers of Barth and Schneider, including [3]. The question is now recorded as Problem 906 [4]. If polynomials are admitted it is immediate, so we treat the transcendental form. Write $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $B(c, R) = \{z \in \mathbb{C} : |z - c| < R\}$.

For a nonempty open set $U \subset \mathbb{C}$ and an entire function f, define

$$A_U(f) = \{n \in \mathbb{N} : f^{(n)} \text{ has a zero in } U\}.$$

The following elementary reformulation isolates the quantifier that the problem requires.

LEMMA 1.1. *For an entire function f, the following assertions are equivalent.*

(1) *For every strictly increasing sequence $(n_k)_{k\geq 0}$ in $\mathbb{N}$, the set*

$$\bigcup_{k\geq 0} \{z \in \mathbb{C} : f^{(n_k)}(z) = 0\}$$

is dense in $\mathbb{C}$.

(2) *For every nonempty open set $U \subset \mathbb{C}$, the set $A_U(f)$ is cofinite in $\mathbb{N}$, that is, $\mathbb{N} \setminus A_U(f)$ is finite.*

Proof. Assume (2). Fix a strictly increasing sequence (n_k) and a nonempty open set U. Since $A_U(f)$ is cofinite there is an integer $N(U)$ such that $n \geq N(U)$ implies $n \in A_U(f)$. The sequence (n_k) is unbounded, so $n_k \geq N(U)$ for some k, and then $f^{(n_k)}$ has a zero in U. The displayed union therefore meets every nonempty open set and is dense.

Conversely, suppose that (2) fails. Then for some nonempty open set U the set $\mathbb{N} \setminus A_U(f)$ is infinite; enumerate it increasingly as (n_k). Each $f^{(n_k)}$ is zero-free in U, so the union of their zero sets misses U and is not dense. □

Theorem 1.2. *There exists a transcendental entire function f such that, for every nonempty open set $U \subset \mathbb{C}$, there is an integer $N(U)$ for which*

$$n \geq N(U) \quad \Longrightarrow \quad f^{(n)} \text{ has a zero in } U.$$

The same function can be chosen to have order exactly two and to satisfy

$$|f(z)| \leq \sqrt{2}\exp(|z|^2), \qquad z \in \mathbb{C}. \tag{1}$$

Consequently, the zero sets indexed by every strictly increasing sequence of derivative orders have dense union.

The proof below shows that the selected function has order exactly two, while (1) gives type at most one. The Boas–Reddy consequence, with its source quantifier made explicit, is recorded in Section 4.

MacLane [13] constructed an entire function whose successive derivatives are dense in the space of entire functions, and Barth and Schneider [3] prescribed discrete zero sets along one constructed sequence of derivative orders. Neither result yields the cofinite condition in Lemma 1.1. Boas and Reddy's expanded article [6] contained a further positive Theorem 2, but their erratum withdrew its construction and left that conclusion open [7].

On 25 April 2026, Almeida first posted a detailed proposed solution and later uploaded a manuscript using the planar Gaussian entire function [2, 1]. Later that day Chojecki posted a second Gaussian-route note; on 1 June he clarified that it was intended as a full solution and reiterated that Almeida had posted first [8]. These posts precede the present manuscript. The author learned of them after completing and circulating this proof.

Almeida's manuscript and the present work share the cofinite reformulation, the Fock normalization $1/\sqrt{k!}$ and a final Borel–Cantelli argument. Almeida uses Gaussian coefficients, the Edelman–Kostlan expected zero measure, Laguerre asymptotics and an Offord-type estimate. Chojecki's forum description likewise invokes the cofinite reformulation and the Edelman–Kostlan–Offord Gaussian route. The contribution here is an alternative bounded-coefficient proof with a different local hole-probability mechanism and a deterministic finite-type growth bound, the latter yielding the Boas–Reddy counterexample.

Here is the mechanism. At each point, condition on all coefficients except the saddle-selected one; planar area then gives an exponential lower-tail bound for the derivative. Averaging a measurable nonnegative deficit around a circle and using the mean-value identity on the zero-free event turns the strict circular-mean gain of the radial potential into a deficit of order $\sqrt{n}$, hence a summable fixed-disk hole probability. Borel–Cantelli [10, Section 2.3] and a countable disk basis then produce one sample. For background on random series, see Kahane [12]. As accessed 16 August 2026, the online record still displays OPEN while recording claimed solutions and warning that relevant literature may be missing [4, 2, 8].

2. A bounded random Fock series

Let $\overline{\mathbb{D}}$ be the closed unit disk and let $m_{\mathbb{D}}$ be normalised planar Lebesgue measure on $\overline{\mathbb{D}}$. Put

$$\Omega = \overline{\mathbb{D}}^{\mathbb{N}}, \qquad \mathcal{F} = \mathcal{B}(\overline{\mathbb{D}})^{\otimes\mathbb{N}}, \qquad \mathbb{P} = m_{\mathbb{D}}^{\otimes\mathbb{N}},$$

and equip Ω with the product topology. A countable product of compact metric spaces is compact and metrizable, and $\mathcal{F}$ is the Borel σ-algebra of Ω. For $\omega = (\omega_k)_{k\geq 0} \in \Omega$ set $\xi_k(\omega) = \omega_k$. The coordinate variables $\xi_0, \xi_1, \ldots$ are then independent and uniformly distributed on $\overline{\mathbb{D}}$. Define

$$f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}. \tag{2}$$

The bound $|\xi_k| \leq 1$ gives normal convergence on compact subsets of $\mathbb{C}$, uniformly in ω. Hence every sample defines an entire function, term by term differentiation is legitimate, and for $n \geq 0$,

$$F_n(z) := f^{(n)}(z) = \sum_{m=0}^{\infty} \xi_{n+m} \frac{\sqrt{(n+m)!}}{m!}\, z^m. \tag{3}$$

The partial sums of (3) are continuous on $\Omega \times \mathbb{C}$ and converge uniformly on $\Omega \times \{|z| \leq M\}$ for every M: the absolute-value majorant has successive-term ratio $M\sqrt{n+m+1}/(m+1) \to 0$. Hence $(\omega, z) \mapsto F_n(\omega, z)$ is jointly continuous; in particular $\omega \mapsto F_n(\omega, z)$ is $\mathcal{F}$-measurable for each fixed z. The law $m_{\mathbb{D}}$ has no atoms, so $\mathbb{P}\{\xi_k = 0\} = 0$ for every k, and therefore

$$\Omega_0 = \{\omega \in \Omega : \xi_k(\omega) \neq 0 \text{ for every } k \geq 0\} \tag{4}$$

has $\mathbb{P}(\Omega_0) = 1$. On Ω_0 all Taylor coefficients in (2) are nonzero, so f is not a polynomial.

The bounded coefficients also give the growth bound. By Cauchy–Schwarz with weights $2^{k/2}$,

$$\begin{aligned} |f(z)| &\leq \sum_{k\geq 0} \frac{|z|^k}{\sqrt{k!}} \\ &\leq \left(\sum_{k\geq 0} \frac{2^k |z|^{2k}}{k!}\right)^{1/2} \left(\sum_{k\geq 0} 2^{-k}\right)^{1/2} = \sqrt{2}\exp(|z|^2). \end{aligned}$$

Thus (1) holds for every $\omega \in \Omega$. Put

$$\Omega_2 = \{\omega : |\xi_k(\omega)| \geq \tfrac{1}{2} \text{ for infinitely many } k\}.$$

Independence and $\mathbb{P}\{|\xi_k| < 1/2\} = 1/4$ give $\mathbb{P}(\Omega_2) = 1$. For $\omega \in \Omega_0 \cap \Omega_2$, the coefficient formula for order gives

$$\rho(f) = \limsup_{k\to\infty} \frac{k \log k}{\frac{1}{2}\log(k!) + \log(1/|\xi_k|)}.$$

The growth bound gives $\rho(f) \leq 2$. Along the subsequence on which $|\xi_k| \geq 1/2$, the displayed quotient tends to two because $\log(k!) \sim k \log k$. Hence every sample in $\Omega_0 \cap \Omega_2$ has order exactly two and type at most one.

For $r \geq 0$ write

$$a_{n,m}(r) = \frac{\sqrt{(n+m)!}}{m!}\, r^m, \qquad S_n(r) = \sum_{m=0}^{\infty} a_{n,m}(r),$$

and set

$$p_n(z) = \tfrac{1}{2}\log(n!) + |z|\sqrt{n}, \qquad L_n = 2 + \log(n+1). \tag{5}$$

We use the extended-real convention $\log 0 = -\infty$. Every occurrence of L_n below is under hypotheses that force $n \geq 1$.

We shall use the elementary estimate

$$\log(m!) \leq m\log m - m + \log(m+1) + 1, \qquad m \geq 1.$$

It follows from $\sum_{j=1}^{m} \log j \leq \int_1^{m+1} \log t\, dt$ and $m\log(1 + 1/m) \leq 1$. Its loss relative to Stirling's estimate is logarithmic, and only $L_n = o(\sqrt{n})$ is used below.

Lemma 2.1 (Saddle estimates). *Let $n \geq 1$.*

(1) *If $r \geq 0$, $1 \leq r\sqrt{n}$ and $r \leq \sqrt{n}$, then, for $m = \lfloor r\sqrt{n} \rfloor$,*

$$\log a_{n,m}(r) \geq \tfrac{1}{2}\log(n!) + r\sqrt{n} - 2 - \log(n+1);$$

by (5) the right-hand side equals $p_n(z) - L_n$ for every z with $|z| = r$.

(2) *For every $r \geq 0$,*

$$\log S_n(r) \leq \tfrac{1}{2}\log(n!) + r\sqrt{n} + \frac{r + r^2}{2}.$$

Consequently, if $|z| \leq R$ then

$$\log|F_n(z)| - p_n(z) \leq \frac{R + R^2}{2} \tag{6}$$

for every $\omega \in \Omega$, with the inequality read trivially when $F_n(z) = 0$.

PROOF. Factor $a_{n,m}(r) = \sqrt{n!}\, b_{n,m}(r)$, where

$$b_{n,m}(r) = \frac{\sqrt{(n+1)(n+2)\cdots(n+m)}}{m!}\, r^m,$$

which is legitimate because $(n+m)! = n!\,(n+1)(n+2)\cdots(n+m)$.

For (1) put $x = r\sqrt{n}$. Each of the m factors under the square root is at least n, so $b_{n,m}(r) \geq n^{m/2} r^m/m! = x^m/m!$. For $m = \lfloor x \rfloor$ the hypotheses give $1 \leq m \leq x \leq n$, since $x \geq 1$ and $x = r\sqrt{n} \leq \sqrt{n}\sqrt{n} = n$. By the preceding estimate,

$$\log \frac{x^m}{m!} \geq m\log x - m\log m + m - \log(m+1) - 1 = m\log\frac{x}{m} + m - \log(m+1) - 1.$$

Here $m\log(x/m) \geq 0$ because $m \leq x$; also $m > x - 1$ and $\log(m+1) \leq \log(n+1)$ because $m \leq n$. Hence

$$\log\frac{x^m}{m!} \geq (x-1) - \log(n+1) - 1 = x - 2 - \log(n+1),$$

and adding $\frac{1}{2}\log(n!)$ to both sides gives (1); the identification of the right-hand side with $p_n(z) - L_n$ is the definition (5).

For (2), note that

$$b_{n,m}(r) = \sqrt{\frac{\binom{n+m}{n}}{m!}}\, r^m.$$

If $r > 0$, put $t = r/(r + \sqrt{n}) \in (0,1)$. Then

$$\frac{\binom{n+m}{n} r^{2m}}{m!} = \left(\binom{n+m}{n} t^m\right) \frac{(r^2/t)^m}{m!}.$$

Cauchy–Schwarz, the binomial series $\sum_{m\geq 0}\binom{n+m}{n}t^m = (1-t)^{-(n+1)}$ and the exponential series give

$$\begin{aligned}\sum_{m\geq 0} b_{n,m}(r) &\leq \left(\sum_{m\geq 0}\binom{n+m}{n}t^m\right)^{1/2}\left(\sum_{m\geq 0}\frac{(r^2/t)^m}{m!}\right)^{1/2}\\ &= (1-t)^{-(n+1)/2}\exp\left(\frac{r^2}{2t}\right).\end{aligned}$$

Put $u = r/\sqrt{n}$. Since $\log(1+u) \leq u$ for $u \geq 0$ and $\sqrt{n} \geq 1$,

$$\begin{aligned}\frac{n+1}{2}\log(1+u) + \frac{r(r+\sqrt{n})}{2} &\leq \frac{(n+1)r}{2\sqrt{n}} + \frac{r^2 + r\sqrt{n}}{2}\\ &= r\sqrt{n} + \frac{r}{2\sqrt{n}} + \frac{r^2}{2}\\ &\leq r\sqrt{n} + \frac{r + r^2}{2}.\end{aligned}$$

Adding $\frac{1}{2}\log(n!)$ gives (2) for $r > 0$; the case $r = 0$ is immediate because $S_n(0) = \sqrt{n!}$. Finally $|\xi_{n+m}| \leq 1$, so (3) and the triangle inequality give $|F_n(z)| \leq S_n(|z|)$. Applying the bound just proved with $r = |z|$, and using that $r \mapsto (r + r^2)/2$ is nondecreasing, gives (6) whenever $|z| \leq R$. □

The second ingredient is anti-concentration. If ξ is uniform on $\overline{\mathbb{D}}$ then, for $a \neq 0$, $w \in \mathbb{C}$ and $\varepsilon \geq 0$,

$$\mathbb{P}\{|a\xi + w| < \varepsilon\} \leq \left(\frac{\varepsilon}{|a|}\right)^2, \tag{7}$$

because the set $\{\zeta : |a\zeta + w| < \varepsilon\}$ is the disk of centre $-w/a$ and radius $\varepsilon/|a|$, whose normalised area is at most $(\varepsilon/|a|)^2$.

Lemma 2.2 (Pointwise logarithmic tail). *Let $n \geq 1$ and let $z \in \mathbb{C}$ satisfy $1 \leq |z|\sqrt{n}$ and $|z| \leq \sqrt{n}$. Then, for every $s \geq 0$,*

$$\mathbb{P}\{p_n(z) - \log|F_n(z)| > s + L_n\} \leq e^{-2s}. \tag{8}$$

Proof. Put $m = \lfloor |z|\sqrt{n} \rfloor$ and $k = n + m$, and split (3) by isolating the coefficient ξ_k:

$$F_n(\omega, z) = a\,\xi_k(\omega) + w(\omega),$$

where

$$a = \frac{\sqrt{(n+m)!}}{m!}\, z^m$$

is deterministic and

$$w(\omega) = \sum_{j\neq m} \xi_{n+j}(\omega)\frac{\sqrt{(n+j)!}}{j!}\, z^j$$

is the remainder. We first check that w is finite and measurable. The series defining w converges absolutely for every ω, since $|\xi_{n+j}| \leq 1$ and $\sum_{j\geq 0} a_{n,j}(|z|) = S_n(|z|) < \infty$ by Lemma 2.1(2); so w is finite everywhere, not merely almost surely. It is also a measurable function of ω, being a uniform limit of continuous functions on Ω. The independence needed below comes from the fact that w does not depend on the coordinate ω_k: it is measurable with respect to $\mathcal{G} = \sigma(\xi_j : j \neq k)$, since the index $j = m$ is omitted from the sum. Note also that $|z| \geq 1/\sqrt{n} > 0$, so $a \neq 0$ and $|a| = a_{n,m}(|z|)$.

Since $|z| =: r$ satisfies the hypotheses of Lemma 2.1(1), that lemma gives $\log|a| \geq p_n(z) - L_n$. Hence, for $s \geq 0$,

$$\{p_n(z) - \log|F_n(z)| > s + L_n\} = \{|F_n(z)| < e^{p_n(z)-L_n-s}\} \subseteq \{|a\xi_k + w| < |a|e^{-s}\}.$$

Now write $\Omega = \overline{\mathbb{D}}\times\Omega'$, where the first factor carries the coordinate ω_k and $\Omega' = \overline{\mathbb{D}}^{\mathbb{N}\setminus\{k\}}$ carries the remaining coordinates, and correspondingly $\mathbb{P} = m_{\mathbb{D}} \otimes \mathbb{P}'$. Because w is $\mathcal{G}$-measurable it is a function of $\omega' \in \Omega'$ alone, so Tonelli's theorem gives

$$\mathbb{P}\{|a\xi_k + w| < |a|e^{-s}\} = \int_{\Omega'} m_{\mathbb{D}}\{\zeta \in \overline{\mathbb{D}} : |a\zeta + w(\omega')| < |a|e^{-s}\}\, d\mathbb{P}'(\omega').$$

For each fixed ω' the inner measure is at most $(e^{-s})^2 = e^{-2s}$ by (7), with $\varepsilon = |a|e^{-s}$; the bound is uniform in ω', so integrating over Ω' preserves it. This proves (8). □

The pointwise estimate is used through the following circle-average bound.

LEMMA 2.3 (Circle-average upper tail). *Let $Y \geq 0$ be jointly measurable on $\Omega \times [0, 2\pi]$ and suppose that*

$$\mathbb{P}\{Y(\cdot, \theta) > s\} \leq e^{-s/2}, \qquad s \geq 0,$$

for every $\theta \in [0, 2\pi]$. Thus, for each fixed θ, the map $\omega \mapsto Y(\omega, \theta)$ is a nonnegative random variable; ω is the sample point and θ is the angular parameter. Put $\overline{Y}(\omega) = (2\pi)^{-1} \int_0^{2\pi} Y(\omega, \theta)\, d\theta$. Then

$$\mathbb{P}\{\overline{Y} \geq t\} \leq 2e^{-t/8}, \qquad t \geq 8.$$

PROOF. The tail hypothesis and the layer-cake formula give

$$\mathbb{E}Y(\cdot, \theta) = \int_0^\infty \mathbb{P}\{Y(\cdot, \theta) > s\}\, ds \leq 2$$

for every θ. Hence $\mathbb{E}\overline{Y} \leq 2$ by Tonelli's theorem, and $\overline{Y}$ is integrable. Put $E = \{\overline{Y} \geq t\}$ and $q = \mathbb{P}(E)$. Then $t\mathbf{1}_E \leq \overline{Y}\mathbf{1}_E$, and Tonelli's theorem gives

$$tq \leq \mathbb{E}(\overline{Y}\mathbf{1}_E) = \frac{1}{2\pi} \int_0^{2\pi} \mathbb{E}(Y(\cdot, \theta)\mathbf{1}_E)\, d\theta.$$

For every θ, the layer-cake formula and

$$\mathbb{P}(\{Y(\cdot, \theta) > s\} \cap E) \leq \min\{q, e^{-s/2}\}$$

yield

$$\begin{aligned} \mathbb{E}(Y(\cdot, \theta)\mathbf{1}_E) &\leq \int_0^\infty \min\{q, e^{-s/2}\}\, ds \\ &= \int_0^{2\log(1/q)} q\, ds + \int_{2\log(1/q)}^\infty e^{-s/2}\, ds \\ &= 2q \log \frac{e}{q}, \end{aligned}$$

where the computation assumes $0 < q \leq 1$. If $q > 0$ we may cancel q and obtain $t \leq 2\log(e/q) = 2 + 2\log(1/q)$, so $q \leq e^{1-t/2}$. For $t \geq 8$ we have $e^{1-t/2} = e^{1-3t/8}e^{-t/8} \leq e^{-2}e^{-t/8} \leq 2e^{-t/8}$. The case $q = 0$ is immediate. □

3. Zero-free disks and proof of the theorem

We first isolate the geometric gain that drives the estimate. For $c \in \mathbb{C}$ and $r > 0$ set

$$\gamma(c, r) = \frac{1}{2\pi} \int_0^{2\pi} |c + re^{i\theta}|\, d\theta - |c|.$$

LEMMA 3.1 (Strict circular-mean gap). *For all $c \in \mathbb{C}$ and $r > 0$ we have $\gamma(c, r) > 0$.*

Proof. Set

$$\varphi(\theta) = |c + re^{i\theta}| + |c - re^{i\theta}| - 2|c|.$$

The triangle inequality gives $\varphi \geq 0$, and φ is continuous. If $c = 0$, then $\varphi \equiv 2r > 0$. If $c \neq 0$, choose θ_0 with $re^{i\theta_0} = irc/|c|$. The Pythagorean identity gives

$$\varphi(\theta_0) = 2\sqrt{|c|^2 + r^2} - 2|c| > 0.$$

Thus continuity implies $\int_0^{2\pi} \varphi(\theta)\, d\theta > 0$. The substitution $\theta \mapsto \theta + \pi$ gives

$$2\gamma(c, r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\theta)\, d\theta > 0,$$

as required. □

Next we record the mean value step in the exact form needed, with the strict radius margin made explicit.

Lemma 3.2 (Mean value property on a zero-free disk). *Let $c \in \mathbb{C}$ and $\rho > 0$, let G be holomorphic and zero-free on $B(c, \rho)$, and let $0 < r < \rho$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |G(c + re^{i\theta})|\, d\theta = \log |G(c)|.$$

Proof. Since $B(c, \rho)$ is simply connected and G is zero-free there, G has a holomorphic logarithm g on $B(c, \rho)$ [9, Chapter IV]. Thus $\log |G| = \operatorname{Re} g$ is harmonic. The strict inequality $r < \rho$ gives $\overline{B(c, r)} \subset B(c, \rho)$, so the harmonic mean-value property on $B(c, r)$ gives the asserted identity. □

Now fix $c \in \mathbb{Q} + i\mathbb{Q}$ with $c \neq 0$ and a rational R with $0 < R < |c|$, and put

$$r = \frac{R}{2} < R, \tag{9}$$

so that $\overline{B(c, r)} \subset B(c, R)$ with a strict radius margin. On the circle $|z - c| = r$ we have

$$0 < \delta := |c| - r \leq |z| \leq |c| + r =: M,$$

the left inequality because $r < R < |c|$. Consequently there is an integer $n_0 = n_0(c, R) \geq 1$ such that

$$1 \leq \delta\sqrt{n} \quad \text{and} \quad M \leq \sqrt{n} \qquad \text{for all } n \geq n_0, \tag{10}$$

and then the hypotheses of Lemma 2.2 hold at every point of the circle $|z - c| = r$, with the same n_0 for every $\theta \in [0, 2\pi]$.

The next device packages the hole argument: on a zero-free disk, the circular average of p_n gains $\gamma(c, r)\sqrt{n}$ over its centre, whereas the circular average of $\log |F_n|$ equals its central value. We therefore record the nonnegative pointwise shortfall in a real-valued measurable form, without changing it on the zero-free event. Define

$$\ell(w) = \begin{cases} \log |w|, & w \neq 0, \\ 0, & w = 0, \end{cases} \qquad D_{n,z} = \max\{p_n(z) - \ell(F_n(z)) - L_n,\ 0\},$$

and, for $n \geq n_0$,

$$\overline{D}_n = \frac{1}{2\pi}\int_0^{2\pi} D_{n,c+re^{i\theta}}\,d\theta.$$

The function ℓ is Borel measurable on $\mathbb{C}$, and $(\omega, z) \mapsto F_n(\omega, z)$ is jointly continuous, so $(\omega, \theta) \mapsto D_{n,c+re^{i\theta}}(\omega)$ is jointly measurable on $\Omega \times [0, 2\pi]$ and $\overline{D}_n$ is a measurable function of ω.

Because $\ell(w) \geq \log|w|$ for every $w \in \mathbb{C}$ in the extended-real order, with equality unless $w = 0$, we have, for $s \geq 0$,

$$\{D_{n,z} > s\} \subseteq \{p_n(z) - \log|F_n(z)| > s + L_n\}.$$

Combining this with Lemma 2.2 and using $e^{-2s} \leq e^{-s/2}$ for $s \geq 0$ gives the tail hypothesis of Lemma 2.3 explicitly: for $n \geq n_0$, every z on the circle $|z - c| = r$ and every $s \geq 0$,

$$\mathbb{P}\{D_{n,z} > s\} \leq \mathbb{P}\{p_n(z) - \log|F_n(z)| > s + L_n\} \leq e^{-2s} \leq e^{-s/2}. \tag{11}$$

So Lemma 2.3 applies to $Y(\omega, \theta) = D_{n,c+re^{i\theta}}(\omega)$ and to $\overline{Y} = \overline{D}_n$.

Let

$$H_n(c, R) = \{\omega \in \Omega : F_n(\omega, \cdot) \text{ has no zero in } B(c, R)\}.$$

Lemma 3.3 (Measurability of the hole event). $H_n(c, R) \in \mathcal{F}$.

Proof. For rational $0 < \rho < R$, let

$$Q_\rho = (\mathbb{Q} + i\mathbb{Q}) \cap B(c, \rho), \qquad M_\rho(\omega) = \inf_{z \in Q_\rho} |F_n(\omega, z)|.$$

The set Q_ρ is countable and dense in $\overline{B(c, \rho)}$, including by interior approximation at boundary points. Hence continuity in z gives

$$M_\rho(\omega) = \min_{|z-c|\leq\rho} |F_n(\omega, z)|,$$

while measurability of each evaluation makes M_ρ measurable. Therefore

$$H_n(c, R) = \bigcap_{\substack{\rho\in\mathbb{Q}\\ 0<\rho<R}} \{M_\rho > 0\} \in \mathcal{F}.$$

Indeed, zero-freeness on $B(c, R)$ gives positivity on every compact subdisk, and the converse follows by choosing rational ρ with $|z - c| \leq \rho < R$ for each $z \in B(c, R)$. □

We can now estimate $\mathbb{P}(H_n(c, R))$. Fix $n \geq n_0$ and work on the event $H_n(c, R)$, on which F_n is holomorphic and zero-free on $B(c, R)$ by the definition of that event. Since $r = R/2 < R$ by (9), Lemma 3.2 applies with $G = F_n$ and $\rho = R$ and gives

$$\frac{1}{2\pi}\int_0^{2\pi} \log|F_n(c + re^{i\theta})|\,d\theta = \log|F_n(c)|. \tag{12}$$

Lemma 2.1(2), applied with R replaced by $|c|$ and evaluated at $z = c$, gives

$$\log|F_n(c)| \le p_n(c) + C(c), \qquad C(c) = \frac{|c| + |c|^2}{2}. \tag{13}$$

On $H_n(c,R)$ we have $F_n(z) \neq 0$ for $z \in \overline{B(c,r)}$, so $\ell(F_n(z)) = \log|F_n(z)|$ there and therefore $D_{n,z} \ge p_n(z) - \log|F_n(z)| - L_n$ on the circle $|z - c| = r$. Averaging over that circle and using

$$\frac{1}{2\pi}\int_0^{2\pi} p_n(c + re^{i\theta})\,d\theta = \tfrac{1}{2}\log(n!) + \sqrt{n}\cdot\frac{1}{2\pi}\int_0^{2\pi} |c + re^{i\theta}|\,d\theta = p_n(c) + \gamma(c,r)\sqrt{n},$$

together with (12) and (13), we obtain on $H_n(c,R)$ that

$$\overline{D}_n \ge p_n(c) + \gamma(c,r)\sqrt{n} - \log|F_n(c)| - L_n \ge \gamma(c,r)\sqrt{n} - C(c) - L_n =: T_n,$$

that is,

$$H_n(c,R) \subseteq \{\overline{D}_n \ge T_n\}, \qquad T_n = \gamma(c,r)\sqrt{n} - C(c) - 2 - \log(n+1). \tag{14}$$

By Lemma 3.1 we have $\gamma(c,r) > 0$, and $C(c) + 2 + \log(n+1) = o(\sqrt{n})$, so there is $n_1 = n_1(c,R) \ge n_0$ with

$$T_n \ge \max\Big\{\tfrac{1}{2}\gamma(c,r)\sqrt{n},\ 8\Big\}, \qquad n \ge n_1. \tag{15}$$

For $n \ge n_1$, Lemma 2.3 together with (11), (14) and (15) gives

$$\mathbb{P}(H_n(c,R)) \le \mathbb{P}\{\overline{D}_n \ge T_n\} \le 2e^{-T_n/8} \le 2\exp\left(-\frac{\gamma(c,r)}{16}\sqrt{n}\right). \tag{16}$$

The right-hand side of (16) is summable, since $\exp(-\gamma(c,r)\sqrt{n}/16) \le n^{-2}$ for all large n, and the finitely many terms with $n < n_1$ are each at most 1. Hence

$$\sum_{n=0}^{\infty} \mathbb{P}(H_n(c,R)) < \infty. \tag{17}$$

All the events involved are measurable by Lemma 3.3, so the first Borel–Cantelli lemma [10, Section 2.3] applies and shows that

$$\mathbb{P}\Big(\limsup_{n\to\infty} H_n(c,R)\Big) = 0; \tag{18}$$

that is, almost surely only finitely many F_n are zero-free in $B(c,R)$.

Proof of Theorem 1.2. Let $\mathcal{B}$ be the family of disks $B(c,R)$ with $c \in \mathbb{Q} + i\mathbb{Q}$, $c \neq 0$, $R \in \mathbb{Q}_{>0}$ and $R < |c|$; it is countable. We check that every nonempty open set $U \subset \mathbb{C}$ contains a member of $\mathcal{B}$. Since U is open and nonempty it contains a point $z_0 \neq 0$, and there is $\varepsilon > 0$ with $B(z_0,\varepsilon) \subset U$ and $\varepsilon < |z_0|/2$. Choose $c \in \mathbb{Q} + i\mathbb{Q}$ with $|c - z_0| < \varepsilon/4$; then $|c| \ge |z_0| - \varepsilon/4 > 0$, so $c \neq 0$. Choose a rational R with $0 < R < \min\{\varepsilon/2, |c|\}$.

Then $R < |c|$, and every z with $|z - c| < R$ satisfies $|z - z_0| < \varepsilon/4 + \varepsilon/2 < \varepsilon$, so $B(c, R) \subset B(z_0, \varepsilon) \subset U$ and $B(c, R) \in \mathcal{B}$.

By (18), for each $B = B(c, R) \in \mathcal{B}$ there is a $\mathbb{P}$-null set $\mathcal{N}_B$ outside of which only finitely many F_n are zero-free in B. The union $\mathcal{N} = \bigcup_{B \in \mathcal{B}} \mathcal{N}_B$ is a countable union of null sets and is therefore null. With Ω_0 as in (4) and Ω_2 as above, we have $\mathbb{P}((\Omega_0 \cap \Omega_2) \setminus \mathcal{N}) = 1$; in particular this event is nonempty. Fix $\omega \in (\Omega_0 \cap \Omega_2) \setminus \mathcal{N}$ and let f be the entire function (2) determined by it.

Then f is transcendental, because $\omega \in \Omega_0$ makes every Taylor coefficient of f nonzero; moreover $\omega \in \Omega_2$ makes its order exactly two, and f satisfies (1). Let $U \subset \mathbb{C}$ be nonempty and open and choose $B \in \mathcal{B}$ with $B \subset U$. Since $\omega \notin \mathcal{N}_B$, only finitely many $F_n = f^{(n)}$ are zero-free in B. Let $N(U)$ be one more than the largest such index, or let $N(U) = 0$ if there is no such index. Then every $n \geq N(U)$ gives a zero of $f^{(n)}$ in B, hence in U. This is the first assertion of Theorem 1.2, and the last assertion follows from it by Lemma 1.1. □

4. A consequence for a theorem of Boas and Reddy

Under their standing assumption that f is transcendental, Boas and Reddy assert that if f is of order at most two and finite type, then f admits arbitrarily large disks on which infinitely many successive derivatives are zero-free [5, Theorem 1]. They clarify that one fixed disk serves infinitely many derivatives [5, p. 65]. Their expanded article also stated a distinct Theorem 2 [6]. Their published erratum addresses Theorem 2 [7]. It withdraws the supporting construction and says that conclusion remains open.

Corollary 4.1. *The function in Theorem 1.2 is a counterexample to the assertion of [5, Theorem 1] as printed.*

Proof. Let f be the function supplied by Theorem 1.2. The estimate (1) places f in the growth class assumed by Boas and Reddy. For every fixed nonempty disk D, Theorem 1.2 supplies an integer $N(D)$ such that $f^{(n)}$ has a zero in D whenever $n \geq N(D)$. Consequently only the finitely many derivatives with $n < N(D)$ can be zero-free in D. Thus no disk, of any radius, is zero-free for infinitely many successive derivatives of f, contrary to the printed conclusion. □

Acknowledgement

The author thanks Xi Chen for detailed suggestions that improved the introduction, theorem formulation, order calculation and circle-average estimate.

Declarations

Use of AI. In July–August 2026, the author used OpenAI GPT-5.6 Sol and Anthropic Claude Fable 5 to assist with the small-ball estimate and editing, source retrieval, manuscript revision and independent argument auditing. The author checked every

AI-assisted mathematical and bibliographic claim and takes sole responsibility for the mathematical content and exposition of this article.
Competing interests. The author declares no competing interests. *Funding.* The author received no specific funding for this work. *Data availability.* No data were generated or analysed in this theoretical study.

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